A regularized arithmetic Riemann–Roch theorem via metric degeneration

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Abstract

Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be an arithmetic surface whose complex fiber $\mathcal{X}_{\mathbb{C}}$ is isometric to a hyperbolic Riemann surface $\overline{\Gamma\backslash\mathbb{H}}$ without elliptic points. Let $\overline{\mathcal{S}}_{k+1}$ be a hermitian line bundle on $\mathcal{X}$ such that the induced complex line bundle S_{k+1} is isometric to the line bundle of cusp forms of weight $2(k+1)$ on $\overline{\Gamma\backslash\mathbb{H}}$. Both the hyperbolic metric on $\overline{\Gamma\backslash\mathbb{H}}$ and the Petersson metric on S_{k+1} are singular. The main resul of the thesis is a regularized arithmetic Riemann–Roch theorem for $\overline{\mathcal{S}}_{k+1}$ valid up to an implicit constant.
The proof proceeds by metric degeneration: We regularize the metrics under consideration in an ϵ-neighborhood of the singularities, then we apply the arithmetic Riemann–Roch of Gillet and Soulé, and finally we let the parameter ϵ go to zero. Both sides of the formula blow up through metric degeneration. On one side the exact asymptotic expansion is computed from the definition of the smooth arithmetic intersection numbers.
The divergent term on the other side is the ζ-regularized determinant of the Laplacian acting on 1-forms with values in S_{k+1} associated to the ϵ-regularized metrics. We first define and compute a regularization of the determinant of the corresponding Laplacian associated to the singular metrics, which will later occur in the regularized arithmetic Riemann–Roch theorem. To do so we adapt and generalize ideas of Jorgenson–Lundelius, D'Hoker–Phong and Sarnak.
Then, we prove a formula for the on-diagonal heat kernel associated to the hermitian line bundle $\overline{S}_{k+1}$ on a model cusp, from which its behavior close to a cusp is transparent. This expression is related to an expansion in terms of the eigenfunctions associated to the Whittaker equation, which we prove in an appendix. Further estimates of the heat kernel associated to the hermitian line bundle $\overline{S}_{k+1}$ on the surface $\overline{\Gamma\backslash\mathbb{H}}$ prove the main theorem.

Zusammenfassung

Es sei $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ eine arithmetische Fläche, deren komplexe Faser $\mathcal{X}_{\mathbb{C}}$ isometrisch zu einer hyperbolischen Riemannschen Fläche $\overline{\Gamma \backslash \mathbb{H}}$ ohne elliptische Punkte ist. Weiter sei $\overline{\mathcal{S}}_{k+1}$ ein hermitisches Geradenbündel auf $\mathcal{X}$, so dass das induzierte komplexe Geradenbündel S_{k+1} isometrisch zum Geradenbündel der Spitzenformen von Gewicht $2(k+1)$ auf $\overline{\Gamma \backslash \mathbb{H}}$ ist. Sowohl die hyperbolische Metrik auf $\overline{\Gamma \backslash \mathbb{H}}$ als auch die Petersson-Metrik auf S_{k+1} sind hier singulär. Das Hauptresultat dieser Arbeit ist ein regulariserter arithmetischer Satz von Riemann–Roch für $\overline{\mathcal{S}}_{k+1}$ gültig bis auf eine implizite Konstante.

Der Beweis des Resultats erfolgt durch metrische Degeneration: Wir regularisieren die betreffenden Metriken in einer ϵ-Umgebung der Singularitäten, wenden dann den arithmetischen Riemann–Roch-Satz von Gillet und Soulé an und lassen schließlich den Parameter ϵ gegen Null gehen. Durch die metrische Degeneration entsteht auf beiden Seiten der Formel ein divergenter Term. Die asymptotische Entwicklung der Divergenz berechnet sich auf der einen Seite direkt aus der Definition der glatten arithmetischen Selbstschnittzahlen.

Der divergente Term auf der anderen Seite ist die ζ-regularisierte Determinante des zu den ϵ-regulariserten Metriken assoziiert Laplace-Operators, der auf den 1-Formen mit Werten in S_{k+1} operiert. Wir definieren und berechnen zuerst eine Regularisierung des entsprechenden zu den singulären Metriken assoziierten Laplace-Operators; diese wird später im regulariserten Riemann–Roch-Satz auftauchen. Zu diesem Zweck passen wir Ideen von Jorgenson–Lundelius, D'Hoker–Phong und Sarnak auf die vorliegende Situation an und verallgemeinern diese.

Schließlich beweisen wir eine Formel für den zum hermiteschen Geradenbündel $\overline{S}_{k+1}$ assoziierten Wärmeleitungskern auf der Diagonalen bei einer Modellspitze. Diese Darstellung steht im Zusammenhang mit einer Entwicklung nach zur Whittaker-Gleichung assoziierten Eigenfunktionen, die im Anhang bewiesen wird. Weitere Abschätzungen des zum hermitischen Geradenbündel $\overline{S}_{k+1}$ gehörigen Wärmeleitungskerns auf $\overline{\Gamma \backslash \mathbb{H}}$ schließen den Beweis des Hauptresultats ab.

Contents

Introduction

Arakelov theory, initiated by Arakelov in [3], is an arithmetic intersection theory whose main idea is to "compactify" arithmetic objects adding hermitian metrics on the induced complex counterparts. The arithmetic Riemann–Roch theorem is a statement in Arakelov theory that the computes the arithmetic degree of the determinant of cohomology of a hermitian vector bundle. Different procedures to attach a hermitian metric to the determinant of cohomology have been proposed: Faltings defines what is now known as Faltings' metric in [21], and proves a first version of the theorem for hermitian line bundles on surfaces equipped with a smooth metric. A different procedure, introduced by Quillen in [53], is used for the version of the theorem proven by Gillet and Soulé, with fundamental contributions of Bismut, in [27, 28, 29]. The latter theorem is valid in higher dimensions and also if the arithmetic variety is allowed to have singularities away from the generic fiber, but it requires a smooth metric on the archimedean fibers of the bundle and of the variety.

It turns out that for many natural applications of the theorem, such as for modular curves, the assumption of smoothness of the metric is too restrictive. The problem of extending the arithmetic Riemann–Roch theorem, in the version of Gillet–Soulé, to the non-smooth setting has been addressed in the past decade by a few works: In his dissertation [24] Freixas proves an arithmetic Riemann–Roch theorem for a twist of the relative dualizing sheaf on a modular curve whose metric on the complex fiber has cusp singularities, later generalized by himself in [25] to arbitrary powers of this line bundle. The statement follows from a Mumford-type isometry on the moduli space $\mathcal{M}_{g,n}$ of n-punctured Riemann surfaces of genus g. Also in his dissertation [31], Hahn combines results from the regularized spectral theory of Jorgenson and Lundelius [39] with an explicit value of the Selberg's $\mathcal{Z}$-function from Freixas' thesis to obtain an arithmetic Riemann–Roch theorem for the relative dualizing sheaf on a modular curve with cusp singularities. Finally, a recent result of Freixas and von Pippich [26] addresses the orbifold case, covering therefore the situation where torsion points are present, using Mayer–Vietoris type formulae for determinants of elliptic operators.

The aim of this work is to address the case of the bundle of cusp forms of weight $2(k+1)$ on a modular curve equipped with a metric with cusp singularities with a new technique: metric degeneration. The main idea is to regularize the metrics under consideration in a small ϵ-neighborhood of the singularities, then to apply the smooth theorem of Gillet and Soulé, and finally to examine the behavior of its terms in the limit for $\epsilon \to 0$. The advantage of this technique is that it does not rely on the description of the moduli space of the geometric objects involved, the drawback is that it needs accurate information on the heat kernel associated to the hermitian line bundle under consideration.

To illustrate the structure of the thesis we need to briefly introduce the notation necessary to state the smooth arithmetic Riemann–Roch theorem. Let $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ be an arithmetic surface whose fiber at infinity $X = \mathcal{X}_{\mathbb{C}}$ is equipped with a Kähler metric. Let $\overline{\omega}_{\mathcal{X}}$ be its relative dualizing sheaf equipped with the induced hermitian metric, and let $\overline{\mathcal{L}}$ be another smooth hermitian line bundle on $\mathcal{X}$. These metrics induce L^2-metrics on the cohomology spaces with values in $\mathcal{L}$. Let $\det'\left(\Delta^1_{\overline{L}}\right)$ be the determinant of positive eigenvalues of the generalized Laplacian on 1-forms with values in $L = \mathcal{L}_{\mathbb{C}}$, and let δ_f be a contribution of the singular fibers. Finally, denote by $\overline{\mathcal{L}}_1.\overline{\mathcal{L}}_2$ the arithmetic intersection number of the hermitian line bundles $\overline{\mathcal{L}}_1$ and $\overline{\mathcal{L}}_2$. Then

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{L})_{L^2}\right) + \widehat{\deg}\left(\det \mathrm{H}^1(\mathcal{X},\mathcal{L})^\vee_{L^2}\right) + \frac{1}{2}\log\left(\det{}'\left(\Delta^1_{\overline{L}}\right)\right) + \delta_f$$
$$= \frac{1}{12}\left(6\overline{\mathcal{L}}.\overline{\mathcal{L}} - 6\overline{\mathcal{L}}.\overline{\omega}_{\mathcal{X}} + \overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right) + \frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1\left(\overline{\omega}_X\right).$$

We now proceed to describe the structure of this work. In the first chapter we review the smooth intersection theory on arithmetic surfaces, and we state the smooth arithmetic Riemann–Roch theorem of Gillet–Soulé with an example. We follow [59].

If the metrics on X and on L are not smooth, the regularized determinant $\det{}'\left(\Delta^1_{\overline{L}}\right)$ is not defined. In the second chapter we address this issue for the complex hermitian line bundle S_{k+1} of cusp forms of weight $2(k+1)$ for a suitable Fuchsian subgroup Γ equipped with the Petersson metric. Specifically, we generalize the regularization approach of Jorgenson–Lundelius [39] from $k=0$ to $k\geq 0$; defining in this way the regularized determinant of the Laplacian

$$\det{}^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right).$$

Then we explicitly compute the latter quantity in terms of the Selberg's $\mathcal{Z}$-function $\mathcal{Z}_\Gamma(s)$ associated to Γ. This computation in the compact setting has been already discussed in the theoretical physics literature, most notably by D'Hoker and Phong [17, 18]. We present a self-contained proof giving credits where appropriate. The final result of the chapter is the following theorem.

Theorem 2.8.4. *The regularized determinant of the Laplacian* $\Delta^1_{\overline{S}_{k+1}}$ *is given by the expression*

$$\det{}^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right) = \begin{cases} \mathcal{Z}'_\Gamma(1)\, e^{-c_0 \operatorname{vol}_{\mathrm{hyp}}(X)}\, 2^{\frac{\operatorname{vol}_{\mathrm{hyp}}(X)}{6\pi}+2}, & k=0, \\ \mathcal{Z}_\Gamma(k+1)\, e^{-c_k \operatorname{vol}_{\mathrm{hyp}}(X)}\, 2^{\frac{(3k+1)\operatorname{vol}_{\mathrm{hyp}}(X)}{6\pi}}, & k\geq 1. \end{cases}$$

Here $\mathcal{Z}_\Gamma(s)$ *is the Selberg zeta function associated to* Γ, *and*

$$c_k = \frac{\log(G(2k+1))}{2\pi} - \frac{2k-1}{4\pi}\log(\Gamma(2k+1)) + \frac{(2k+1)^2}{8\pi} - \frac{(2k+1)\log(2\pi)}{4\pi} - \frac{\zeta'(-1)}{\pi},$$

where $G(Z)$ *denotes the Barnes* G*-function.*

The fundamental tool for this chapter is an explicit expression for the heat kernel associated to the hyperbolic Laplacian of weight $k+1$ on $\mathbb{H}$ provided by Fay [23].

In the third chapter we consider the heat kernel on the fundamental domain $\mathcal{F}_\infty$ for the action of a cusp stabilizer on $\mathbb{H}$. Using the Poisson summation formula and intricate computations to evaluate the arising Fourier coefficients we find an expression from which the growth of the heat kernel at a cusp can be deduced. It is analogous to a formula proved by Müller [46] for $k=0$ and for a cusp on an arbitrary closed manifold. This formula is stated in the following theorem.

Theorem 3.4.1. *The on-diagonal weight k heat kernel $K_k^\infty(t;z,z)$ on the model cusp has the expression*

$$K_k^\infty(t;z,z)=\sum_{n=1}^{\infty}\frac{1}{4\pi n}\sum_{j=0}^{k-1}\frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\,\Gamma(j+1)}\,W_{k,k-j-\frac{1}{2}}(4\pi ny)^2+y\,\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}}$$
$$+\sum_{n=1}^{\infty}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi^3 n}\int\limits_0^\infty r\sinh(2\pi r)e^{-tr^2}\sum_{\kappa=\pm k}\left|\Gamma\left(\kappa+\frac{1}{2}+ir\right)\right|^2 W_{-\kappa,ir}(4\pi ny)^2\,\mathrm{d}r,$$

where $W_{k,\mu}(Z)$ denotes the Whittaker W-function.

In the fourth chapter we define suitable regularizations of the Petersson and hyperbolic metrics that live in an ϵ-neighborhood of the cusps. We use the subscript ϵ to denote objects associated to these regularized metrics. The only terms of the smooth arithmetic Riemann–Roch theorem of Gillet–Soulé for this regularized setting that do not have a finite limit for $\epsilon\to 0$ are the Quillen correction term, i.e., the regularized determinant of the Laplacian, and the arithmetic intersection numbers. Denoting by p the number of cusps of X, the asymptotic expansion of the intersection numbers is given by the formula

$$\frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\mathcal{S}}_{k+1,\epsilon}-6\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}+\overline{\omega}_{\mathcal{X},\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}\right) \tag{4.2.1}$$
$$=\frac{p}{12}\log(\epsilon)+p\left(\frac{k}{2}+\frac{1}{6}\right)\log(-\log(\epsilon))+O_{\mathcal{X},k}(1).$$

In comparison with the corresponding computation of [26], our proof is more technically involved. The difference arises from using directly the smooth intersection numbers defined by Gillet–Soulé and not the L_1^2-formalism developed by Bost in [9]. Using this formula we define the regularized arithmetic intersection numbers. Moreover, we deduce from it the asymptotic expansion of the determinant of the Laplacian corresponding to the ϵ-metrics.

Corollary 4.3.1. *The asymptotic expansion*

$$\log\left(\det{}'\left(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\right)\right)=\frac{p}{6}\log(\epsilon)+\frac{p\,(3k+1)}{3}\log(-\log(\epsilon))+O_{\Gamma,k}(1)\qquad(\epsilon\to 0)$$

holds.

In the second part of the fourth chapter, we provide partial results on the latter asymptotic expansion from the definition of its left hand side. Let $K_k^\Gamma(t;z,w)$ be the weight k heat kernel on X associated to the Petersson and hyperbolic metrics, $K_{k,\epsilon}^\Gamma$ its counterpart associated to the ϵ-regularized metrics, and $P_1,\ldots,P_j$ the cusps of X. The decomposition of the trace of the ϵ-regularized heat kernel given by

$$
\begin{aligned}
\int\limits_X K^{\Gamma}_{k,\epsilon}(t;z,z)\,\mu_\epsilon(z) = &\int\limits_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)} K^{\Gamma}_{k}(t;z,z)\,\mu_{\mathrm{hyp}}(z) && \text{(a)}\\
&+\int\limits_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K^{\Gamma}_{k,\epsilon}(t;z,z)-K^{\Gamma}_{k}(t;z,z)\right)\mu_{\mathrm{hyp}}(z) && \text{(b)}\\
&+\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} K^{\Gamma}_{k,\epsilon}(t;z,z)\,\mu_\epsilon(z). && \text{(c)}
\end{aligned}
$$

induces a corresponding decomposition of the regularized determinant, which, up to an appropriate renormalization, is given by

$$-\log\left(\det{}'\left(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\right)\right) = \text{(A)}+\text{(B)}+\text{(C)}.$$

Here the letters refer to the corresponding pieces in the decomposition of the trace of the heat kernel. Since $K^{\Gamma}_{k}(t;z,z)$ can be written as a Poincaré series indexed by elements of Γ, we further decompose the first piece (A) into a term (AHyp) carrying the identity and the hyperbolic contribution and a term (APar) carrying the parabolic one. The parabolic part is shown to have the, partially implicit, asymptotic expansion

$$
\begin{aligned}
\text{(APar)} = &-p\left(k+\frac{1}{2}\right)\log(-\log(\epsilon))\\
&+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi^3 n}\int_0^\infty r\sinh(2\pi r)e^{-tr^2}\right.\right.\\
&\left.\left.\times\sum_{\kappa=\pm k}\left|\Gamma\left(\kappa+\frac{1}{2}+ir\right)\right|^2 W_{-\kappa,ir}(4\pi n y)^2\,\mathrm{d}r\frac{\mathrm{d}y}{y^2},s\right)\right)\Bigg|_{s=0}+O(1)\qquad(\epsilon\to 0)\,.
\end{aligned}
$$

This expression accounts for the k-dependent part of the asymptotic in the corollary 4.3.1. On the other hand, the hyperbolic part converges to the hyperbolically regularized determinant defined in the second chapter. Specifically, up to an explicit renormalization, we find

$$\lim_{\epsilon\to 0}\text{(AHyp)} = -\log\left(\det{}^{*}_{\Gamma}\left(\Delta^{1}_{\overline{S}_{k+1}}\right)\right).$$

We finally state a regularized arithmetic Riemann–Roch theorem, proved through metric degeneration, that is valid up to an implicit constant $C(\Gamma,k)$.

Theorem 4.6.1. *Let $f\colon \mathcal{X}\to\mathcal{S}$ be an arithmetic surface such that $X\simeq X(\Gamma)$ for Γ a cofinite torsion-free and discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and let $\overline{\mathcal{S}}_{k+1}$ be a hermitian line bundle on $\mathcal{X}$ such that the induced hermitian complex line bundle $\overline{\mathcal{S}}_{k+1,\mathbb{C}}$ is isometric to the line bundle of cusp forms $\overline{S}_{k+1}$. Then, we have the equality of real numbers*

$$
\begin{aligned}
&\widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet}}\right)+\widehat{\deg}\left(\mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet}}\right)+\frac{1}{2}\log\left(\det{}^{*}_{\Gamma}\left(\Delta^{1}_{\overline{S}_{k+1}}\right)\right)+\delta_f+C(\Gamma,k)\\
&\quad=\frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1}.\overline{\mathcal{S}}_{k+1}-6\overline{\mathcal{S}}_{k+1},\overline{\omega}_{\mathcal{X}}+\overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right)^{*}+\frac{2\zeta'(-1)+\zeta(-1)}{2}\int\limits_X \mathrm{c}_1(\overline{\omega}_X).
\end{aligned}
$$

In the last section of the chapter we discuss the terms (B) and (C) in the special case $k = 0$.

In the first appendix we provide a generalization of the inversion formula for the Kontorovich–Lebedev transform, obtained applying Weyl–Titchmarsh-theory to the Whittaker equation. The formula is given by the next theorem.

Theorem A.2. *Let $g(u) \in \mathcal{C}_0^\infty(\mathbb{R}_{\geq 0})$ and $k \in \mathbb{Z}$, then we have*

$$g(u) = \sum_{j=0}^{k-1} \frac{(2k-2j-1)}{\Gamma(2k-j)\Gamma(j+1)} \frac{W_{k,k-j-\frac{1}{2}}(2u)}{\sqrt{2u}} \int_0^\infty \frac{W_{k,k-j-\frac{1}{2}}(2v)}{\sqrt{2v}} g(v) \frac{\mathrm{d}v}{v}$$
$$+ \frac{1}{\pi^2} \int_0^\infty r \sinh(2\pi r) \left| \Gamma\left(-k + \frac{1}{2} + ir\right) \right|^2 \frac{W_{k,ir}(2u)}{\sqrt{2u}} \int_0^\infty \frac{W_{k,ir}(2v)}{\sqrt{2v}} g(v) \frac{\mathrm{d}v}{v} \, \mathrm{d}r.$$

The clear affinity of the last result with theorem 3.4.1 suggests that it could be used to directly prove a formula for the weight k heat kernel on a cusp over an arbitrary manifold in the spirit of Müller [46].

Chapter 1

Classical intersection theory on arithmetic surfaces

In this chapter we introduce the theory necessary to state the arithmetic Riemann–Roch theorem for arithmetic surfaces. We follow [59], but if a statement or an argument can be simplified by our restriction on the dimension we do so.

1.1 Hermitian line bundles on arithmetic surfaces

Let $\mathcal{S} = \operatorname{Spec}(\mathbb{Z})$ be the spectrum of the ring of integers.

Definition 1.1.1. An *arithmetic variety over* $\mathcal{S}$ is a regular scheme $\mathcal{X}$ together with a projective flat morphism $f\colon \mathcal{X} \to \mathcal{S}$ such that its generic fiber is smooth. An *arithmetic surface* is a 2-dimensional arithmetic variety.

By abuse of notation we sometimes denote the arithmetic surface $f\colon \mathcal{X} \to \mathcal{S}$ simply by $\mathcal{X}$. In the literature there is a plethora of definitions for the notion of arithmetic surface. Our definition matches the one used by Soulé et al. in [59]; in turn, this definition corresponds to the one used by Soulé in [58], but with the requirement that the generic fiber be geometrically irreducible dropped, and it is a stricter definition than the one used by Liu [44]. An important generalization consists in defining arithmetic varieties over general arithmetic rings, this is done, among others, in Gillet–Soulé [27] and Burgos–Kramer–Kühn [12]. For the moment we content ourselves to observe that we do not work in this more general setting because we need a notion of arithmetic degree.

Notation 1.1.2. Let $\mathcal{X}$ be an arithmetic variety over $\mathcal{S}$, and let $\mathcal{X}_{\mathbb{Q}} = \mathcal{X} \times_{\mathcal{S}} \operatorname{Spec}(\mathbb{Q})$ be its generic fiber. We identify $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \times_{\mathcal{S}} \operatorname{Spec}(\mathbb{C})$ with the complex manifold $\mathcal{X}(\mathbb{C})$. Moreover, we set $X := \mathcal{X}_{\mathbb{C}}$ and we call it the complex fiber of $\mathcal{X}$.

The complex fiber X of an arithmetic surface $\mathcal{X}$ over $\mathcal{S}$ is a 1-dimensional projective complex manifold, i.e., it is a compact Riemann surface.

Example 1.1.3. For any number field K, the scheme $\operatorname{Spec}(\mathcal{O}_K)$ equipped with the natural morphism to $\operatorname{Spec}(\mathbb{Z})$ is a 1-dimensional arithmetic variety.

Example 1.1.4. An arithmetic surface in the sense of [41, paragraph 1.1] is an arithmetic surface according to our definition. Specifically, let Γ be a subgroup of finite index in $\mathrm{PSL}_2(\mathbb{Z})$, then the arithmetic surface associated to Γ constructed in [41, paragraph 4.1] is an arithmetic surface according to our definition.

The main class of interesting examples is the one of modular curves, the difficulty is that one has to keep track of the behavior of the singular fibers.

Example 1.1.5. By theorem 3.7.1 combined with corollaries 2.7.2 and 2.7.4 of [40], the modular curves $Y(N)$ and $Y_1(N)$ classifying elliptic curves with a $\Gamma(N)$-structure for $N \geq 3$ and with a $\Gamma_1(N)$-structure for $N \geq 4$, respectively, are finite étale schemes over $\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)$. Their compactification, defined in paragraph 8.6 of loc. cit., is then a smooth arithmetic surface over $\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right) \subset \mathcal{S}$.

For simplicity of exposition we stick to the requirement that an arithmetic surface is defined over the whole of $\mathcal{S}$. Otherwise, if a surface is only defined over $\operatorname{Spec}\left(\mathbb{Z}\left[\frac{1}{N}\right]\right)$, the difference is that the arithmetic degree, definition 1.1.10, is only defined up to $\sum_{p|N} \mathbb{Q} \cdot \log(p)$.

Definition 1.1.6. A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}})$ on the arithmetic surface $\mathcal{X}$ is an algebraic line bundle $\mathcal{L}$ on $\mathcal{X}$ such that the induced complex line bundle $L = \mathcal{L}_{\mathbb{C}}$ on X is equipped with a smooth hermitian metric $\|\cdot\|_{\overline{\mathcal{L}}}$ which is invariant by complex conjugation.

Notation 1.1.7. Let l be a non-zero section of $\mathcal{L}$, then, for $z \in X$, we write

$$\|l\|_{\overline{\mathcal{L}}}(z) = \|l\|_{\overline{L}}(z).$$

Definition 1.1.8. Two hermitian line bundles are *isomorphic* if there exists an algebraic isomorphism between them that induces an isometry of the induced complex hermitian line bundles. We denote by $\widehat{\operatorname{Pic}}(\mathcal{X})$ the group of isomorphism classes of hermitian line bundles on $\mathcal{X}$, where the group operation is given by the tensor product.

Example 1.1.9. Let $\overline{\mathcal{O}}_{\mathcal{X}} = (\mathcal{O}_{\mathcal{X}}, |\cdot|)$ be the trivial bundle on $\mathcal{X}$ equipped with the absolute value norm on the complex fiber of $\mathcal{X}$. Then $\overline{\mathcal{O}}_{\mathcal{X}}$ is a hermitian line bundle on $\mathcal{X}$, called again the trivial line bundle. It is the identity element of $\widehat{\operatorname{Pic}}(\mathcal{X})$.

Definition 1.1.10. The *arithmetic degree* is defined as the map

$$\widehat{\deg}\colon \widehat{\operatorname{Pic}}(\mathcal{S}) \longrightarrow \mathbb{R},$$

given by the assignment

$$\left[\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}}\right] \longmapsto \log\left([\mathcal{L} : \mathbb{Z}]\right) - \log(\|l\|_{\overline{\mathcal{L}}}),$$

where l is any non-zero rational section of $\mathcal{L}$; where we write $\left[\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}}\right]$ for the isomorphism class of $\left(\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}}\right)$.

Notation 1.1.11. Let f be any function defined on $\widehat{\operatorname{Pic}}(\mathcal{S})$. From now on, we write

$$f(\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}}) := f\left(\left[\mathcal{L}, \|\cdot\|_{\overline{\mathcal{L}}}\right]\right).$$

1.2 Arithmetic Chow groups

The aim of this section is to introduce the notion of arithmetic Chow groups. In what follows we will be interested in $\widehat{\mathrm{CH}}^1(\mathcal{S})$, $\widehat{\mathrm{CH}}^1(\mathcal{X})$ and $\widehat{\mathrm{CH}}^2(\mathcal{X})$ for $\mathcal{X}$ an arithmetic surface, thus we restrict the generality of the exposition to d-dimensional arithmetic varieties with $d \leq 2$. For proofs, more details or the statements in a more general setting we refer to [59].

Let $\mathcal{X}$ be a d-dimensional arithmetic variety with $d \leq 2$, then X is a compact $(d-1)$-dimensional complex manifold with $d-1 \leq 1$. Denote by $A^{p,q}(X)$ the vector space of complex valued differential forms of type (p,q) on X; these spaces are trivial for $p, q \geq 2$. If z is a local coordinate, an element $\omega \in A^{p,q}(X)$ can be expressed as

$$\omega(z) = f(z, \bar{z})\,(\mathrm{d}z)^p \wedge (\mathrm{d}\bar{z})^q,$$

where $f(z,\bar{z}) \in \mathcal{C}^\infty(X) = A^{0,0}(X)$. The Dolbeault differentials $\partial\colon A^{p,q}(X) \to A^{p+1,q}(X)$ and $\bar{\partial}\colon A^{p,q}(X) \to A^{p,q+1}(X)$ are locally given by

$$\partial(f(z,\bar{z})\,(\mathrm{d}z)^p \wedge (\mathrm{d}\bar{z})^q) = \partial_z f(z,\bar{z})\,(\mathrm{d}z)^{p+1} \wedge (\mathrm{d}\bar{z})^q,$$

and

$$\bar{\partial}(f(z,\bar{z})\,(\mathrm{d}z)^p \wedge (\mathrm{d}\bar{z})^q) = \partial_{\bar{z}} f(z,\bar{z})\,(\mathrm{d}z)^{p} \wedge (\mathrm{d}\bar{z})^{q+1},$$

respectively. The usual exterior derivative is given by their sum $\mathrm{d} = \partial + \bar{\partial}$.

The space of Schwartz-continuous linear functionals on $A^{d-1-p,d-1-q}(X)$ is denoted by $D^{p,q}(X)$; elements of this space are called *currents*. There is an inclusion map

$$A^{p,q}(X) \hookrightarrow D^{p,q}(X),$$

given by the assignment

$$\omega \longmapsto [\omega],$$

where

$$[\omega](\alpha) := \int_X \omega \wedge \alpha \qquad \left(\alpha \in A^{d-1-p,d-1-q}(X)\right).$$

The differentials ∂, $\bar{\partial}$ and d define maps ∂', $\bar{\partial}'$ and d' from $D^{p,q}(X)$ to $D^{p+1,q}(X)$, $D^{p,q+1}(X)$ and $D^{p+1,q}(X) \oplus D^{p,q+1}(X)$ respectively. For example, if $T \in D^{p,q}(X)$ and $\alpha \in A^{d-1-(p+1),d-1-q}(X)$ we have $(\partial' T)(\alpha) = T(\partial\alpha)$. An application of Stokes' theorem shows

$$[\mathrm{d}\omega] = (-1)^{p+q+1}(\mathrm{d}'[\omega]) \qquad (\omega \in A^{p,q}(X)).$$

Thus, by abuse of notation, we use again the symbols ∂, $\bar{\partial}$ and d for the maps $(-1)^{p+q+1}\partial'$, $(-1)^{p+q+1}\bar{\partial}'$ and $(-1)^{p+q+1}\mathrm{d}'$. We also introduce the operator $\mathrm{d}^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$.

Important currents are the ones associated to irreducible analytic subvarieties of X. Let $i\colon Z \hookrightarrow X$ be a p-codimensional irreducible subvariety; there is an associated current $\delta_Z \in D^{p,p}(X)$ defined by

$$\delta_Z(\alpha) := \int_Z i^*\alpha \qquad \Big(\alpha \in A^{d-1-p,d-1-p}(X)\Big).$$

The definition is extended by linearity to non-irreducible analytic subvarieties of X.

Remark 1.2.1. If X is 1-dimensional and $p = 1$, the subvariety Z is a point of X. In this case $\delta_Z \in D^{1,1}(X)$ and its evaluation at $\alpha \in A^{0,0}(X)$ is $\delta_Z(\alpha) = \alpha(Z)$.

The wedge product of differential forms naturally defines a wedge product on the induced currents. The definition of the wedge product between general currents is a delicate issue, but it simplifies if one of them is induced by a form. Since we do not need the more general case we only define it in the latter situation.

Definition 1.2.2. Let $\omega \in A^{p,p}(X)$ and $g \in D^{q,q}(X)$, then

$$(g \wedge [\omega])(\alpha) := g(\omega \wedge \alpha) \qquad \Big(\alpha \in A^{d-1-p-q,d-1-p-q}(X)\Big).$$

Let Z be a p-codimensional analytic subvariety of X.

Definition 1.2.3. A *Green current for* Z is a current $g_Z \in D^{p-1,p-1}(X)$ such that there exists $\omega_Z \in A^{p,p}(X)$ satisfying

$$\mathrm{dd}^c g_Z + \delta_Z = [\omega_Z].$$

A class of Green currents which plays an important role in the definition of the arithmetic intersection product is the class of currents associated to Green forms of logarithmic type.

Definition 1.2.4. A form $\widetilde{g}_Z \in A^{0,0}(X \setminus Z)$ is said to be a *Green form of logarithmic type* for Z if, for each local coordinate z centered at an irreducible component of Z, there exists a ∂- and $\bar{\partial}$-closed smooth $(0,0)$-form α and a smooth $(0,0)$-form β such that

$$\widetilde{g}_Z = \alpha \log |z|^2 + \beta,$$

and $[\widetilde{g}_Z]$ is a Green current for Z.

Theorem 1.2.5 (Poincaré–Lelong formula)**.** *Let* $(L, \|\cdot\|)$ *be a hermitian holomorphic line bundle on* X*, let* l *be a non-zero meromorphic section of* L *and* $\mathrm{c}_1(L, \|\cdot\|)$ *the first Chern form of* L*, defined as in [66, paragraph 3.3]. Then* $-\log \|l\|^2 \in L^1(X)$*, hence it induces a current* $[-\log \|l\|^2] \in D^{0,0}(X)$*. This is a Green current for* $\mathrm{div}(l)$*, since we have*

$$\mathrm{dd}^c[-\log \|l\|^2] + \delta_{\mathrm{div}(l)} = [\mathrm{c}_1(L, \|\cdot\|)]\,, \qquad \square$$

Now we translate the theory expressed so far for the complex manifold X to the arithmetic variety $\mathcal{X}$. We start with the definition of differential forms and currents on $\mathcal{X}$.

Definition 1.2.6. Let F be the anti-linear involution on X induced by the complex conjugation. The space of differential forms of type (p,p) on $\mathcal{X}$ is

$$A^{p,p}(\mathcal{X}) := \{\omega \in A^{p,p}(X) \,|\, \omega \text{ real}, F^*(\omega) = (-1)^p \omega\},$$

and the space of currents of type (p,p) on $\mathcal{X}$ is

$$D^{p,p}(\mathcal{X}) := \{T \in D^{p,p}(X) \,|\, T \text{ real}, F^*(T) = (-1)^p T\}.$$

The mapping $\omega \mapsto [\omega]$ defining the embedding $A^{p,p}(X) \to D^{p,p}(X)$ induces an embedding $A^{p,p}(\mathcal{X}) \to D^{p,p}(\mathcal{X})$. Let $Z^p(\mathcal{X})$ be the free abelian group generated by the closed integral p-codimensional subschemes of $\mathcal{X}$, its elements are called cycles. Any cycle $\mathcal{Z} = \sum_j \mathcal{Z}_j \in Z^p(\mathcal{X})$ induces a current $\delta_Z = \sum_j \delta_{Z_j} \in D^{p,p}(\mathcal{X})$, where $Z_j = \mathcal{Z}_{j,\mathbb{C}}$ are the complex fibers of the closed integral p-codimensional subschemes $\mathcal{Z}_j$ and $Z = \mathcal{Z}_{\mathbb{C}}$ is the complex fiber of $\mathcal{Z}$.

Definition 1.2.7. A *Green current for the cycle* $\mathcal{Z} \in Z^p(\mathcal{X})$ is a current $g_Z \in D^{p-1,p-1}(\mathcal{X})$, for $Z = \mathcal{Z}_{\mathbb{C}}$, such that there exists a form $\omega_Z \in A^{p,p}(\mathcal{X})$ for which

$$\mathrm{dd}^c g_Z + \delta_Z = [\omega_Z].$$

Definition 1.2.8. The *group* $\widehat{Z}^p(\mathcal{X})$ *of* p*-codimensional arithmetic cycles in* $\mathcal{X}$ has as elements pairs of the form $(\mathcal{Z}, g_Z)$, where $\mathcal{Z} \in Z^p(\mathcal{X})$ and g_Z is a Green current for $\mathcal{Z}$. The addition is defined componentwise.

Let $\mathcal{Z}$ be a $(p-1)$-codimensional closed integral subscheme with generic point z, i.e., $\overline{\{z\}} = \mathcal{Z}$. Let $j \colon \mathcal{Z} = \overline{\{z\}} \hookrightarrow \mathcal{X}$ be the embedding of $\mathcal{Z}$ in $\mathcal{X}$. Then $f \in k(z)^\times$ induces a meromorphic L^1-function $f_{\mathbb{C}}$ on the complex fiber $Z = \mathcal{Z}_{\mathbb{C}}$, which defines a current $[-\log|f|^2] \in D^{p-1,p-1}(\mathcal{X})$ by

$$[-\log|f|^2](\alpha) := \int_Z -\log|f_{\mathbb{C}}|^2 \wedge j^*\alpha \qquad \Big(\alpha \in A^{d-p,d-p}(\mathcal{X})\Big).$$

By the Poincaré–Lelong formula

$$\mathrm{dd}^c[-\log|f|^2] + \delta_{\mathrm{div}(f_{\mathbb{C}})} = 0;$$

therefore, the quantity $[-\log|f|^2]$ is a Green current for $\mathrm{div}(f)$.

Definition 1.2.9. Let $\widehat{R}^p(\mathcal{X})$ be the subgroup of $\widehat{Z}^p(\mathcal{X})$ generated by pairs of the form $(\mathrm{div}(f), [-\log|f|^2])$, where f is a non-zero rational function on a $(p-1)$-codimensional integral closed subscheme $\mathcal{Z}$ of $\mathcal{X}$. The *p-codimensional arithmetic Chow group of* $\mathcal{X}$ is the quotient

$$\widehat{\mathrm{CH}}^p(\mathcal{X}) := \widehat{Z}^p(\mathcal{X}) / \widehat{R}^p(\mathcal{X}).$$

We underscore that if $\mathcal{X}$ has dimension larger than 2 this definition has to be modified.

Remark 1.2.10. Although this definition proceeds along the lines of the original one given by Arakelov in [3], and it is thus historically the first to appear, it is not the only possible way to define arithmetic Chow rings. Specifically, a different approach, which relates Green currents to Deligne–Beilinson cohomology, has been pioneered by Gillet–Soulé [29], and it has been extended first by Burgos [11] and later by Burgos, Kramer and Kühn [12].

Before turning to the definition of an arithmetic intersection product, we observe that we associated to an arithmetic variety $\mathcal{X}$ two kinds of groups, on the one hand a group of line bundles equipped with hermitian metrics $\widehat{\mathrm{Pic}}(\mathcal{X})$, and on the other hand a set of groups of cycles and associated analytic objects $\widehat{\mathrm{CH}}^p(\mathcal{X})$. As it is the case for their purely geometric counterparts, they can be related.

Definition 1.2.11. The *first arithmetic Chern class* is defined as the map

$$\widehat{\mathrm{c}}_1 \colon \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{X}),$$

given by the assignment

$$[\mathcal{L}, \|\cdot\|] \longmapsto \left[\mathrm{div}(l), [-\log \|l\|^2_{\mathcal{L}}]\right],$$

where l is any non-zero rational section of $\mathcal{L}$.

Proposition 1.2.12. *For any arithmetic variety $\mathcal{X}$, the first arithmetic Chern class*

$$\widehat{\mathrm{c}}_1 \colon \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{X})$$

is an isomorphism. □

The inverse of the first arithmetic Chern class has the expression

$$\widehat{\mathrm{c}}_1^{-1}\left([\mathcal{Z}, g_Z]\right) = \left[\mathcal{O}_{\mathcal{X}}(\mathcal{Z}), \|\cdot\|_Z\right],$$

where $\|f\|^2_Z := |f|^2\, e^{-g_Z}$.

Notation 1.2.13. As it is customary in the literature, by abuse of notation we still denote by $\widehat{\deg}$ the composition $\widehat{\deg} \circ \widehat{\mathrm{c}}_1^{-1} \colon \widehat{\mathrm{CH}}^1(\mathcal{S}) \to \mathbb{R}$.

Defining the intersection product for 1-codimensional arithmetic cycles is particularly simple because of the following corollary to the last proposition.

Corollary 1.2.14. *Let $\mathcal{Z} \in Z^1(\mathcal{X})$ be 1-codimensional cycle and g_Z an associated Green current. Then $g_Z = [\widetilde{g}_Z]$, where $\widetilde{g}_Z$ is a Green form of logarithmic type for Z.* □

Example 1.2.15. We explicitly compute the group $\widehat{\mathrm{CH}}^1(\mathcal{S}) \simeq \widehat{\mathrm{Pic}}(\mathcal{S})$ for $\mathcal{S} = \mathrm{Spec}(\mathbb{Z})$. First let us observe that $Z^1(\mathcal{S}) = \bigoplus_{p \text{ prime}} \mathbb{Z} \cdot p$, and that a Green current for $\mathcal{Z} \in Z^1(\mathcal{S})$ is any real number $g_Z \in \mathbb{R}$. Moreover, the only 0-codimensional point of $\mathrm{Spec}(\mathbb{Z})$ is $\{(0)\}$, and $k((0))^\times/\{\pm 1\} = \mathbb{Q}^\times/\{\pm 1\} \simeq Z^1(\mathcal{S})$, where the isomorphism is given by $\prod_{p \text{ prime}} p^{n_p} \mapsto \sum_{p \text{ prime}} n_p \cdot p$. Thus, for any $[\mathcal{Z}, g_Z] \in \mathrm{CH}^1(\mathcal{S})$ there exists $f_Z \in \mathbb{Q}^\times/\{\pm 1\}$ such that $[\mathcal{Z}, g_Z] = [0, g_Z + \log|f_Z|]$ and the map $[\mathcal{Z}, g_Z] \mapsto g_Z + \log|f_Z|$ is an isomorphism $\widehat{\mathrm{CH}}^1(\mathcal{S}) \xrightarrow{\simeq} \mathbb{R}$. Now let $\left[\overline{\mathcal{L}}\right] \in \widehat{\mathrm{Pic}}(\mathcal{S})$ and l be a generator for $\mathcal{L}$ as a $\mathbb{Z}$-module, then $\left[\overline{\mathcal{L}}\right] \mapsto -2\log\|l\|_{\overline{\mathcal{L}}}$ gives an isomorphism $\widehat{\mathrm{Pic}}(\mathcal{S}) \simeq \mathbb{R}$. The factor 2 has been introduced to let the given isomorphisms $\widehat{\mathrm{CH}}^1(\mathcal{S}) \simeq \mathbb{R}$ and $\widehat{\mathrm{Pic}}(\mathcal{S}) \simeq \mathbb{R}$ correspond to each other via $\widehat{\mathrm{c}}_1 \colon \widehat{\mathrm{Pic}}(\mathcal{S}) \xrightarrow{\simeq} \widehat{\mathrm{CH}}^1(\mathcal{S})$.

1.3 Definition of the arithmetic intersection product

In this section we introduce an intersection product on elements of the Chow group of an arithmetic surface $\widehat{\mathrm{CH}}^*(\mathcal{X}) = \bigoplus_{p=0}^{2} \widehat{\mathrm{CH}}^p(\mathcal{X})$. Due to the limited generality needed for our exposition, we only define the pairing

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \otimes \widehat{\mathrm{CH}}^1(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^2(\mathcal{X}).$$

For the definition of the arithmetic intersection product in broader generality we refer to [59].

Definition 1.3.1. Let $\mathcal{X}$ be an arithmetic surface, and consider the following groups of cycles and the associated Chow groups

$$\begin{aligned} Z^p_{\mathrm{fin}}(\mathcal{X}) &= \{\mathcal{Z} \in Z^p(\mathcal{X}) \,|\, \mathrm{supp}(\mathcal{Z}) \cap \mathcal{X}_{\mathbb{Q}} = \emptyset\}, \\ \mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}) &= Z^p_{\mathrm{fin}}(\mathcal{X}) / \langle \mathrm{div}(f) \rangle, \end{aligned}$$

where f runs over the set of rational functions on $(p-1)$-codimensional integral subschemes not contained in $\mathcal{X}_{\mathbb{Q}}$.

$$\begin{aligned} Z^p_{\mathcal{Y}}(\mathcal{X}) &= \{\mathcal{Z} \in Z^p(\mathcal{X}) \,|\, \mathrm{supp}(\mathcal{Z}) \subseteq \mathcal{Y}\}, \\ \mathrm{CH}^p_{\mathcal{Y}}(\mathcal{X}) &= Z^p_{\mathcal{Y}}(\mathcal{X}) / \langle \mathrm{div}(f) \rangle, \end{aligned}$$

where $\mathcal{Y} \subseteq \mathcal{X}$ is a closed subscheme, and f runs over the set of rational functions on $(p-1)$-codimensional integral subschemes not contained in $\mathcal{Y}$.

$$\widehat{Z}^p(\mathcal{X}_{\mathbb{Q}}) = \{(\mathcal{Z}, g_Z) \,|\, \mathcal{Z} \in Z^p(\mathcal{X}_{\mathbb{Q}}),\ g_Z \text{ is a Green current for } \mathcal{Z}\}.$$

Remark 1.3.2. We have the natural decomposition

$$Z^p(\mathcal{X}) = Z^p_{\mathrm{fin}}(\mathcal{X}) \oplus Z^p(\mathcal{X}_{\mathbb{Q}}).$$

Moreover, let $\mathcal{Z} \subseteq \mathcal{X}$ be a closed subscheme such that $\mathrm{codim}_{\mathcal{X}_{\mathbb{Q}}}(\mathcal{Z}_{\mathbb{Q}}) = p$, there is a natural map

$$\mathrm{CH}^p_{\mathcal{Z}}(\mathcal{X}) \longrightarrow \mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}) \oplus Z^p_{\mathcal{Z}_{\mathbb{Q}}}(\mathcal{X}_{\mathbb{Q}}).$$

We will make use of the following non-trivial geometric result.

Theorem 1.3.3. *Let $\mathcal{Y}, \mathcal{Z} \subseteq \mathcal{X}$ be closed subschemes, then there exists an intersection pairing*

$$\mathrm{CH}^1_{\mathcal{Y}}(\mathcal{X}) \times \mathrm{CH}^1_{\mathcal{Z}}(\mathcal{X}) \longrightarrow \mathrm{CH}^2_{\mathcal{Y} \cap \mathcal{Z}}(\mathcal{X}).$$

We denote by $[\mathcal{Y}' \cdot \mathcal{Z}'] \in \mathrm{CH}^2_{\mathcal{Y} \cap \mathcal{Z}}(\mathcal{X})$ the image of the pair $([\mathcal{Y}'], [\mathcal{Z}']) \in \mathrm{CH}^1_{\mathcal{Y}}(\mathcal{X}) \times \mathrm{CH}^1_{\mathcal{Z}}(\mathcal{X})$. □

Using this theorem we define the geometric part of the arithmetic intersection product. Let $[\mathcal{Y}, g_Y], [\mathcal{Z}, g_Z] \in \widehat{\mathrm{CH}}^1(\mathcal{X})$, we can assume that $\mathcal{Y}$ and $\mathcal{Z}$ are irreducible elements and then extend the definition by linearity. Moreover, by [59, Remark III.2.3.2], since the cycles are 1-codimensional the moving lemma holds. Thus, we can assume without loss of generality that $\mathcal{Y}$ and $\mathcal{Z}$ are properly intersecting cycles. By definition, the elements $[\mathcal{Y}] \in \mathrm{CH}^1_{\mathcal{Y}}(\mathcal{X})$ and $[\mathcal{Z}] \in \mathrm{CH}^1_{\mathcal{Z}}(\mathcal{X})$ are well-defined, and, by theorem 1.3.3, also the class $[\mathcal{Y} \cdot \mathcal{Z}] \in \mathrm{CH}^2_{\mathcal{Y} \cap \mathcal{Z}}(\mathcal{X})$ is well-defined. Thus, using the second map in remark 1.3.2, we find well-defined elements $[\mathcal{Y} \cdot \mathcal{Z}]_{\mathrm{fin}} \in \mathrm{CH}^2_{\mathrm{fin}}(\mathcal{X})$ and $[\mathcal{Y} \cdot \mathcal{Z}]_{\mathbb{Q}} \in Z^2_{(\mathcal{Y} \cap \mathcal{Z})_{\mathbb{Q}}}(\mathcal{X}_{\mathbb{Q}}) \subset Z^2(\mathcal{X}_{\mathbb{Q}})$. To complete the definition of the arithmetic intersection product it remains to define a Green current $g_{Y \cdot Z}$ for the cycle $[\mathcal{Y} \cdot \mathcal{Z}]_{\mathbb{Q}}$.

Definition 1.3.4. Let $g_Y = [\widetilde{g}_Y]$ and g_Z be Green currents for $\mathcal{Y}$ and $\mathcal{Z} \in Z^1(\mathcal{X})$, respectively. Moreover, let ω_Y be a smooth form satisfying

$$\mathrm{dd}^c g_Y + \delta_Y = [\omega_Y].$$

Then their *$*$-product* is defined by the assignment

$$g_Y * g_Z = [\widetilde{g}_Y] \wedge \delta_Z + [\omega_Y] \wedge g_Z.$$

Remark 1.3.5. The $*$-product $g_Y * g_Z$ is a well-defined current in $D^{1,1}(\mathcal{X})$.

In the general situation we would have to prove that the considered $*$-product is not only a current but a Green current for the cycle $\mathcal{Y} \cdot \mathcal{Z}$, i.e., identifying $Y \cdot Z = (\mathcal{Y} \cdot \mathcal{Z})_{\mathbb{C}}$ we would have to show that there exists $\omega \in A^{2,2}(\mathcal{X})$ such that

$$\mathrm{dd}^c(g_Y * g_Z) + \delta_{Y \cdot Z} = [\omega].$$

Since the generic fiber is 1-dimensional, the set of 2-codimensional cycles is empty and the free abelian group $Z^2(\mathcal{X}_{\mathbb{Q}})$ constructed on them is the trivial group. In particular, $Y \cdot Z = \emptyset$ and $\delta_{Y \cdot Z} \equiv 0$. Moreover, $\mathrm{dd}^c(g_Y * g_Z) \in D^{2,2}(\mathcal{X}) = \{0\}$. Thus, the relation above is trivial.

Lemma 1.3.6. *There is an isomorphism*

$$(\mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}) \oplus \widehat{Z}^p(\mathcal{X}_{\mathbb{Q}}))/\langle \widehat{\mathrm{div}}(f) \rangle \xrightarrow{\simeq} \widehat{\mathrm{CH}}^p(\mathcal{X}),$$

given by the assignment

$$[[\mathcal{Y} \cdot \mathcal{Z}]_{\mathrm{fin}}, (0, g_Y * g_Z)] \longmapsto [[\mathcal{Y} \cdot \mathcal{Z}], g_Y * g_Z],$$

where $\widehat{\mathrm{div}}(f) = \big(\mathrm{div}(f), [-\log|f|^2]\big)$, *and f runs over elements of $k(z)^\times$ for $(p-1)$-codimensional points z contained in $\mathcal{X}_{\mathbb{Q}}$.* □

Definition 1.3.7. Let $\mathcal{X}$ be an arithmetic surface, then the *intersection product* is the morphism

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \otimes \widehat{\mathrm{CH}}^1(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^2(\mathcal{X}),$$

given by the assignment

$$([\mathcal{Y}, g_Y], [\mathcal{Z}, g_Z]) \longmapsto [[\mathcal{Y} \cdot \mathcal{Z}], g_Y * g_Z].$$

The well-definedness of the arithmetic intersection product follows from the fact that it factorizes through the morphism

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) \otimes \widehat{\mathrm{CH}}^1(\mathcal{X}) \longrightarrow (\mathrm{CH}^2_{\mathrm{fin}}(\mathcal{X}) \oplus \widehat{Z}^2(\mathcal{X}_{\mathbb{Q}}))/\langle \widehat{\mathrm{div}}(f) \rangle,$$

given by the assignment

$$([\mathcal{Y}, g_Y], [\mathcal{Z}, g_Z]) \longmapsto [[\mathcal{Y} \cdot \mathcal{Z}]_{\mathrm{fin}}, (0, g_Y * g_Z)],$$

and the map of lemma 1.3.6.

Remark 1.3.8. In a more general setting, the intersection product on the arithmetic Chow groups of an arithmetic variety $\mathcal{X}$ of any dimension is a pairing of the form

$$\widehat{\mathrm{CH}}^p(\mathcal{X}) \times \widehat{\mathrm{CH}}^q(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(\mathcal{X}),$$

and, possibly after base changing, it induces a ring structure on the arithmetic Chow group

$$\widehat{\mathrm{CH}}^*(\mathcal{X}) = \oplus_{p=0}^{\dim(\mathcal{X})} \widehat{\mathrm{CH}}^p(\mathcal{X}).$$

Using the ring structure on $\widehat{\mathrm{CH}}^*(\mathcal{X})$ we can extend the first Chern form to the Chern character. Once again, since we only consider line bundles, our situation is simpler than the general one; and our statements have to be modified for higher rank vector bundles.

Definition 1.3.9. The *arithmetic Chern character* is the group morphism

$$\widehat{\mathrm{ch}} \colon \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{X}),$$

defined by

$$\widehat{\mathrm{ch}}\left(\overline{\mathcal{L}}\right) = \exp(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})).$$

Notation 1.3.10. For an element $\alpha \in \widehat{\mathrm{CH}}^*(\mathcal{X})$, let $\alpha^{(p)}$ be the projection of α to $\widehat{\mathrm{CH}}^p(\mathcal{X})$. Also, $\widehat{\mathrm{ch}}^{(p)}$ is the morphism given by the composition of $\widehat{\mathrm{ch}}$ with the projection on $\widehat{\mathrm{CH}}^p(\mathcal{X})$.

An arithmetic Todd class can be defined in the same way.

Definition 1.3.11. The *arithmetic Todd class*

$$\widehat{\mathrm{td}} \colon \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^*(\mathcal{X}),$$

is given by

$$\widehat{\mathrm{td}}(\overline{\mathcal{L}}) := \mathrm{td}(\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})),$$

where $\mathrm{td}(x)$ is the formal power series

$$\mathrm{td}(x) = \frac{xe^x}{e^x - 1} = 1 + \frac{x}{2} + \frac{x^2}{12} + \ldots \in \mathbb{Q}[[x]].$$

Remark 1.3.12. The arithmetic Chow ring of an arithmetic surface $\mathcal{X}$ has the form $\widehat{\mathrm{CH}}^*(\mathcal{X}) = \oplus_{p=0}^2 \widehat{\mathrm{CH}}^p(\mathcal{X})$, thus, for $\overline{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$,

$$\begin{aligned} \widehat{\mathrm{ch}}(\overline{\mathcal{L}}) &= 1 + \widehat{\mathrm{c}}_1(\overline{\mathcal{L}}) + \frac{\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^2}{2}, \\ \widehat{\mathrm{td}}(\overline{\mathcal{L}}) &= 1 + \frac{\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})}{2} + \frac{\widehat{\mathrm{c}}_1(\overline{\mathcal{L}})^2}{12}. \end{aligned}$$

The assignment of an arithmetic Chow ring to an arithmetic surface, and in general to an arithmetic variety, enjoys some functorial properties. We only present one of them in the special case of an arithmetic surface.

Theorem 1.3.13. *Let $f\colon \mathcal{X} \to \mathcal{S}$ be an arithmetic surface. Then there is a pushforward map*

$$f_*\colon \widehat{\mathrm{CH}}^2(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{S}),$$

given by the assignment

$$[\mathcal{Z}, g_Z] \longmapsto [f_*\mathcal{Z}, f_* g_Z],$$

where the pushforward of a 2-codimensional integral closed subscheme $\mathcal{Z}$ with generic point z is given by the assignment

$$f_*(\mathcal{Z}) := [k(z) : k(f(z))]\, \overline{\{f(z))\}},$$

and it is extended by linearity on the non irreducible components, and the pushforward of the current g_Z is given by the assignment

$$(f_* g_Z)(\alpha) := g_Z(f^*\alpha) \qquad \left(\alpha \in A^{0,0}(\mathcal{S})\right). \qquad \square$$

We remark that the definition of the pushforward morphism has to be refined if $\dim(\mathcal{X}) > 2$. Now we associate to the datum of two hermitian line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$ a real number $\overline{\mathcal{L}}.\overline{\mathcal{M}}$, called their arithmetic intersection number. This number synthesizes information on their arithmetic intersection product, and appears in the upcoming arithmetic Riemann–Roch theorem.

Definition 1.3.14. Let $f\colon \mathcal{X} \to \mathcal{S}$ be an arithmetic surface, and $\overline{\mathcal{L}}, \overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ be two hermitian line bundles. Their *arithmetic intersection number* is the real number defined by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := \widehat{\deg}\left(\widehat{c}_1^{\,-1}\left(f_*\left(\widehat{c}_1(\overline{\mathcal{L}}) \cdot \widehat{c}_1(\overline{\mathcal{M}})\right)\right)\right).$$

The next explicit formula for the intersection product is obtained by unraveling the definition.

Proposition 1.3.15. *Let $f\colon \mathcal{X} \to \mathcal{S}$ be an arithmetic surface, $\overline{\mathcal{L}}, \overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ two hermitian line bundles and l, m respective rational sections. Then their arithmetic intersection number has the expression*

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} = \left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\mathrm{fin}} + \left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\infty},$$

where

$$\left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\mathrm{fin}} = (\mathrm{div}(l), \mathrm{div}(m))_{\mathrm{fin}}$$

is the usual geometric intersection of $\mathrm{div}(l)$ *and* $\mathrm{div}(m)$*, and*

$$\left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\infty} = -\log \|l\|_{\overline{L}}(\mathrm{div}(m)) + \int_X \log \|m\|_{\overline{M}}\, \mathrm{c}_1(\overline{L}). \qquad \square$$

Since it is not transparent from the notation, we remark that, while the quantities $\left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\mathrm{fin}}$ and $\left(\overline{\mathcal{L}}, \overline{\mathcal{M}}\right)_{\infty}$ depend on the choice of sections l and m, the intersection number $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ does not.

1.4 Determinant of cohomology and Quillen metric

Given an arithmetic surface $f\colon \mathcal{X}\to\mathcal{S}$, we constructed three sides of the diagram

$$\begin{tikzcd} \widehat{\mathrm{Pic}}(\mathcal{X}) \arrow[r, "\widehat{\mathrm{ch}}^{(2)}"] & \widehat{\mathrm{CH}}^2(\mathcal{X}) \arrow[d, "f_*"] \\ \widehat{\mathrm{Pic}}(\mathcal{S}) \arrow[r, "\widehat{\mathrm{ch}}^{(1)}"] & \widehat{\mathrm{CH}}^1(\mathcal{S}) \end{tikzcd} .$$

The goal of this section is to define a suitable morphism

$$f_!\colon \widehat{\mathrm{Pic}}(\mathcal{X})\to\widehat{\mathrm{Pic}}(\mathcal{S}),$$

while in the next one the commutativity of the constructed diagram will be discussed.

Definition 1.4.1. The *determinant of cohomology* $\lambda(\mathcal{L}) \in \mathrm{Pic}(\mathcal{S})$ is defined by the formula

$$\lambda(\mathcal{L}) = \bigotimes_{q=0,1} \left(\det \mathrm{R}^q f_* \mathcal{L}\right)^{(-1)^q},$$

where $\mathrm{R}^q f_*$ are the right derived functors of f_*, and, by abuse of notation, we denote by V^{-1} the dual $V^\vee$ of V.

Remark 1.4.2. Since $\mathcal{S}$ is affine,

$$\lambda(\mathcal{L}) = \det \mathrm{H}^0(\mathcal{X}, \mathcal{L}) \otimes \det \mathrm{H}^1(\mathcal{X}, \mathcal{L})^\vee.$$

Moreover, the complex vector space $\lambda(\mathcal{L})_{\mathbb{C}}$ is given by

$$\lambda(\mathcal{L})_{\mathbb{C}} \simeq \det \mathrm{H}^0(X, L) \otimes \det \mathrm{H}^1(X, L)^\vee.$$

To complete this definition to a morphism $\widehat{\mathrm{Pic}}(\mathcal{X})\to\widehat{\mathrm{Pic}}(\mathcal{S})$ we need to produce a hermitian metric on the complex component of the determinant of cohomology. Recall that differential forms of type $(0,1)$ with values in L are given by

$$A^{0,1}(X,L) = A^{0,1}(X) \otimes_{A^{0,0}(X)} A^{0,0}(X,L).$$

The differential $\bar{\partial}$ induces a differential $\bar{\partial}_L\colon A^{0,0}(X,L)\to A^{0,1}(X,L)$. Indeed let U be an open set with local coordinate z, the element $l \in A^{0,0}(X,L)$ is mapped via the local isomorphism of L with the trivial bundle to f_l, then we define $\bar{\partial}_L l$ to be the preimage under the same isomorphism of $\bar{\partial}\,(f_l)$. One verifies that this local definition extends to the desired global map. The Dolbeault cohomology, computed from the Dolbeault complex

$$\{0\} \longrightarrow A^{0,0}(X,L) \xrightarrow{\bar{\partial}_L} A^{0,1}(X,L) \longrightarrow \{0\},$$

agrees with the above considered coherent cohomology of the line bundle L on X. Therefore, to define a metric on the determinant of cohomology we will equivalently define a metric on the spaces of differential forms on X with values in L. To define such metric we make the important assumption that the Riemann surface X is equipped with a Kähler metric.

Assumption 1.4.3. The Riemann surface X is equipped with a Kähler metric, i.e., a hermitian metric h on the holomorphic tangent bundle T_X.

Notation 1.4.4. For any local coordinate z, we denote by

$$\mu(z) = \frac{i}{2\pi} h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

the associated volume form.

The hermitian metric h on T_X induces a hermitian metric on $A^{0,1}(X)$. Tensoring this metric with the hermitian metric on L gives a hermitian scalar product $\langle s(z), t(z)\rangle$ for elements s, t of $A^{0,1}(X,L)$.

Definition 1.4.5. Let s, t be elements of $A^{0,0}(X,L)$ or $A^{0,1}(X,L)$; their L^2-*product* is defined by the formula

$$\langle s, t\rangle_{L^2} = \int_X \langle s(z), t(z)\rangle \mu(z),$$

where the pointwise scalar product are the one induced by the choice of the hermitian metric on L and the one defined above, respectively.

To let the L^2-metric on $A^{0,0}(X,L)$ and $A^{0,1}(X,L)$ induce a metric on $\mathrm{H}^0(X,L)$ and $\mathrm{H}^1(X,L)$, respectively, we need that every class in one of the two cohomology groups has a unique harmonic representative, i.e., we need to apply Hodge theorem.

Remark 1.4.6. The differential $\bar{\partial}_L$ has an adjoint $\bar{\partial}_L^*\colon A^{0,1}(X,L)\to A^{0,0}(X,L)$ with respect to the L^2-metric.

Definition 1.4.7. We define the *Laplacians*

$$\Delta^0_{\overline{L}} = \bar{\partial}^*_{\overline{L}} \bar{\partial}_L \qquad \text{acting on } A^{0,0}(X, L),$$

and

$$\Delta^1_{\overline{L}} = \bar{\partial}_L \bar{\partial}^*_{\overline{L}} \qquad \text{acting on } A^{0,1}(X, L).$$

Both these Laplacians are generalized Laplacians in the sense of definition 2.2 of [7]. Hodge theory gives canonical identifications

$$\mathrm{H}^0(X, L) \simeq \ker\left(\Delta^0_{\overline{L}}\right) \subseteq A^{0,0}(X, L), \qquad \mathrm{H}^1(X, L) \simeq \ker\left(\Delta^1_{\overline{L}}\right) \subseteq A^{0,1}(X, L).$$

Definition 1.4.8. The metrics on the $\mathbb{C}$-vector spaces $\mathrm{H}^0(X, L)$ and $\mathrm{H}^1(X, L)$ obtained via these isomorphisms naturally induce a metric on their respective determinant, and in turn on $\lambda(\mathcal{L})$. This is the L^2*-metric on the determinant of cohomology* $\lambda(\mathcal{L})$.

Remark 1.4.9. The determinant of cohomology equipped with the L^2-metric is a hermitian holomorphic line bundle on $\mathcal{S}$. We write

$$\lambda(\overline{\mathcal{L}})_{L^2} := \left[\lambda(\overline{\mathcal{L}}), \|\cdot\|_{L^2}\right] \in \widehat{\mathrm{Pic}}(\mathcal{S}).$$

By a more general point of view the content of the last remark is quite accidental. Indeed, on arithmetic varieties of higher dimension the L^2-metric, which is defined fiberwise, is in general not smooth or even continuous moving along the base variety. A suitable correction for the L^2-metric in relative dimension 1 has been given by Quillen [53], and the corrected metric has been named Quillen metric after him. Besides being a hermitian metric on the determinant of cohomology, which is the motivation that led Quillen to define it in the first place, the Quillen metric carries arithmetic information, and it is therefore the right notion of metric on the determinant of cohomology in our setting. We now introduce it.

Definition 1.4.10. The *heat kernel* $K_{\overline{L}}(t; z, w)$ is a family of sections of $(L)_z \otimes (L^\vee)_w$ depending on $t \in \mathbb{R}_{>0}$ and satisfying the following properties

(1) It is $\mathcal{C}^1$ in the time variable t and $\mathcal{C}^2$ in the space variables z, w.

(2) Denote by $\Delta^0_{\overline{L},z}$ the laplacian $\Delta^0_{\overline{L}}$ acting on the variable z, then

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \Delta^0_{\overline{L},z}\right) K_{\overline{L}}(t; z, w) = 0.$$

(3) For any section l of L with compact support

$$\lim_{t \to 0} \int_X K_{\overline{L}}(t; z, w)\, l(w)\, \mu(w) = l(z).$$

Observe that we do not need a compact surface for this definition to be make sense.

Remark 1.4.11. In comparison to definition 2.15 of [7], the notion of bundle of half-densities does not enter our definition of heat kernel. Indeed our fixed volume form induces a canonical choice of a global section of the bundle of 1-densities.

Assumption 1.4.12. From now on, we assume the Laplacian $\Delta^0_{\overline{L}}$ to be symmetric and positive semi-definite.

By the assumption, and using the compactness of X, we are in the hypothesis of proposition 2.36 of [7] and subsequent comments, which we quote without proof.

Proposition 1.4.13. *The Laplacian $\Delta^0_{\overline{L}}$ has discrete non-negative spectrum $\{\lambda_j\}_{j\geq 0}$. Thus, the heat kernel is unique and is given by*

$$K_{\overline{L}}(t; z, w) = \sum_{j\geq 0} e^{-\lambda_j t}\varphi_j(z)\,\widetilde{\varphi}_j(w),$$

where $\varphi_j(z)$ is the eigenfunction associated to the eigenvalue λ_j, and $\widetilde{\varphi}_j(w)$ is the image of $\varphi_j(w)$ via the canonical isomorphism $L \simeq L^\vee$ induced by the L^2-inner product. □

This remark implies that the on-diagonal heat kernel $K_{\overline{L}}(t; z, z)$ is non-negative and monotonically non-increasing in t. We write $\mathcal{M}(f(t), s)$ for the transform of the function $f(t)$, discussed in appendix B.

Definition 1.4.14. The *trace of the heat kernel* $K_{\overline{L}}(t; z, w)$ is

$$\operatorname{Tr} K_{\overline{L}}(t) := \int_X K_{\overline{L}}(t; z, z)\mu(z) \qquad (t \in \mathbb{R}_{>0})\,.$$

Since X is compact, the trace $\operatorname{Tr} K_{\overline{L}}(t)$ is convergent for any $t > 0$.

Definition 1.4.15. Let

$$N_{\overline{L}} = \lim_{t\to\infty} \operatorname{Tr} K_{\overline{L}}(t)$$

be the *number of zero modes* of $\overline{L}$. The *spectral zeta function* associated to $\Delta^0_{\overline{L}}$ is given by the formula

$$\zeta_{\overline{L}}(s) = \frac{1}{\Gamma(s)}\mathcal{M}\left(\operatorname{Tr} K_{\overline{L}}(t) - N_{\overline{L}}, s\right) \qquad (\operatorname{Re}(s) \gg 1)\,.$$

Moreover, the *regularized determinant of the Laplacian* is

$$\det{}'\left(\Delta^0_{\overline{L}}\right) = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\overline{L}}(s)\right)_{s=0}.$$

The well-definedness of the the regularized determinant, i.e., the holomorphicity of the spectral zeta function at $s = 0$, is not an immediate result. We point the reader to theorem V.1.1 of [59] for a proof. We also observe that Hodge theory implies

$$N_{\overline{L}} = \dim \mathrm{H}^0(X, L),$$

because $N_{\overline{L}}$ is the multiplicity of the zero eingevalue of $\Delta^0_{\overline{L}}$.

Remark 1.4.16. The name determinant is justified by the following observation. Let us assume $\mathrm{Re}(s) \gg 1$, then the spectral zeta function can be rewritten as

$$\zeta_{\overline{L}}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\int_X K_{\overline{L}}(t; z, z)\mu(z) - N_{\overline{L}} \right) t^{s-1} dt = \sum_{j>0} \lambda_j^{-s}.$$

Therefore, we have the purely formal relation

$$\det{}' \left(\Delta^0_{\overline{L}} \right) = \prod_{j>0} \lambda_j.$$

Also, since X is compact, the number $N_{\overline{L}}$ equals the multiplicity of the zero eigenvalue of $\Delta^0_{\overline{L}}$. Let ω_X be the holomorphic contangent bundle of X, then $N_{\overline{\omega}_X}$ is the number of connected components of X. Finally, we observe that the prime in the notation of the determinant means that we are removing the zero modes from the trace of the heat kernel.

Observation 1.4.17. Let us consider the scaled Laplacian $c\,\Delta^0_{\overline{L}}$, where $c \in \mathbb{R}_{>0}$. Its determinant scales according to the relation

$$\det{}' \left(c\,\Delta^0_{\overline{L}} \right) = c^{\zeta_{\overline{L}}(0)} \det{}' \left(\Delta^0_{\overline{L}} \right).$$

Definition 1.4.18. The *Quillen metric* on the complex part $\lambda(\mathcal{L})_{\mathbb{C}}$ of the determinant of cohomology is given by the formula

$$h_Q = h_{L^2} \cdot \det{}' \left(\Delta^1_{\overline{L}} \right)^{-1}.$$

The Quillen correction term $\det{}' \left(\Delta^1_{\overline{L}} \right)^{-1}$ is the inverse of the square of the analytic torsion of Ray–Singer [54]. We denote by $\lambda(\overline{\mathcal{L}})_Q \in \widehat{\mathrm{Pic}}(\mathcal{S})$ the class of the determinant of cohomology of $\mathcal{L}$ equipped with the Quillen metric at every complex fiber, then, the desired map

$$f_! \colon \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{S})$$

is given by the assignment

$$f_! \left(\overline{\mathcal{L}} \right) = \lambda(\overline{\mathcal{L}})_Q.$$

Remark 1.4.19. As discussed in [28, page 27], the isomorphism, given by Serre's duality,

$$\lambda(L) \simeq \lambda\left(\omega_X \otimes L^\vee \right)$$

is an isometry for the L^2-metric. Moreover, we have

$$\det{}' \left(\Delta^1_{\overline{L}} \right) = \det{}' \left(\Delta^0_{\overline{\omega}_X \otimes \overline{L}^\vee} \right); \tag{1.4.1}$$

therefore, it is an isometry for the Quillen metric as well. We additionally observe that, in the case of arithmetic relative dimension 1, i.e., when the fibers X have complex dimension 1, the eigenspaces of $\Delta^0_{\overline{L}}$ and $\Delta^1_{\overline{L}}$ are naturally in bijection. This is explained in [59, page 123]. Thus, we conclude

$$\det{}' \left(\Delta^0_{\overline{L}} \right) = \det{}' \left(\Delta^1_{\overline{L}} \right). \tag{1.4.2}$$

1.5 The arithmetic Riemann–Roch theorem

In this section we finally state the arithmetic Riemann–Roch theorem. Let $\omega_{\mathcal{X}}$ be the relative dualizing sheaf of the arithmetic surface $f\colon \mathcal{X}\to\mathcal{S}$. Since f is projective we can factorize it as follows,

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_{\mathcal{S}} \arrow[d, "p"] \\ & \mathcal{S} \end{tikzcd} , \tag{1.5.1}$$

where i is a closed embedding and p the projection from the projective n-space over $\mathcal{S}$. Let $\mathcal{N}_i$ be the normal bundle associated to the immersion of $\mathcal{X}$ in $\mathbb{P}^n_{\mathcal{S}}$ and $i^*\mathcal{T}_p$ the restriction to $\mathcal{X}$ of the relative tangent bundle to p. By definition 6.4.7 and theorem 6.4.32 of [44] the relative dualizing sheaf is isomorphic to the canonical sheaf, and we have

$$\omega_{\mathcal{X}} \simeq \det(i^*\mathcal{T}_p)\otimes\det(\mathcal{N}_i)^{-1}.$$

Moreover, since f is generically smooth, the complex part $\omega_{\mathcal{X},\mathbb{C}}$ is the holomorphic cotangent bundle ω_X of the Riemann surface X, and it inherits naturally a hermitian metric from the Kähler metric on the complex fiber. We have therefore a well-defined element $\overline{\omega}_{\mathcal{X}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$. To state the main theorem we need the notion of Todd genus $\widehat{\mathrm{td}}(f)$ for the morphism f. If f is smooth, it is defined by the arithmetic Todd class $\widehat{\mathrm{td}}(\overline{T}_{\mathcal{X}})$ of the relative tangent bundle, according to definition 1.3.11. If the morphism f is generically smooth but not smooth, the relative dualizing sheaf is not invertible everywhere, and the relative tangent bundle only exists as a virtual object. In this case the definition is technical, and, since it will not be used afterwards, we skip it. It can be found in [29, par. 2.6.2]. Now, let us consider the following diagram, constructed in the previous sections,

$$\begin{tikzcd} \widehat{\mathrm{Pic}}(\mathcal{X}) \arrow[r] \arrow[d] & \widehat{\mathrm{CH}}^2(\mathcal{X}) \arrow[d] \\ \widehat{\mathrm{Pic}}(\mathcal{S}) \arrow[r] & \widehat{\mathrm{CH}}^1(\mathcal{S}) \end{tikzcd} , \tag{1.5.2}$$

where the composition of upper and right arrow is given by the assignment

$$[\overline{\mathcal{L}}] \longmapsto \left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}})\widehat{\mathrm{td}}(f)\right)^{(2)} \longmapsto f_*\left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}})\widehat{\mathrm{td}}(f)\right)^{(1)},$$

and the composition of left and bottom arrow is given by

$$[\overline{\mathcal{L}}] \longmapsto \lambda(\overline{\mathcal{L}})_Q \longmapsto \widehat{\mathrm{c}}_1\left(\lambda(\overline{\mathcal{L}})_Q\right).$$

The arithmetic Riemann–Roch theorem gives an explicit expression for the lack of commutativity of this diagram. Before stating it we introduce some additional complex characteristic classes, whose building block is the usual first Chern class $\mathrm{c}_1\colon \mathrm{Pic}(X)\to\mathrm{H}^2(X,\mathbb{R})$

on line bundles of a complex analytic variety. Also, for a general function g defined on $\mathrm{Pic}(X)$, and L a line bundle on X, we use the notation

$$g(L) := g([L]).$$

Definition 1.5.1. Let X be a Riemann surface and $[L] \in \mathrm{Pic}(X)$. Then the *Chern character* of $[L]$ is

$$\mathrm{ch}(L) := \exp(\mathrm{c}_1(L)) = 1 + \mathrm{c}_1(L) + \frac{\mathrm{c}_1(L)^2}{2} + ... \in \mathrm{H}^*(X, \mathbb{R}),$$

the *Todd class* of L is

$$\mathrm{td}(L) := \mathrm{td}(\mathrm{c}_1(L)) = 1 + \frac{\mathrm{c}_1(L)}{2} + \frac{\mathrm{c}_1(L)^2}{12} + ... \in \mathrm{H}^*(X, \mathbb{R}),$$

and the R-*genus* of $[L]$ is

$$\mathrm{R}(L) := \sum_{m \text{ odd } \geq 1} \left(2\zeta'(-m) + \zeta(-m) \sum_{j=1}^{m} \frac{1}{j} \right) \frac{\mathrm{c}_1(L)^m}{m!} \in \mathrm{H}^*(X, \mathbb{R}).$$

Remark 1.5.2. The smooth morphism $f_{\mathbb{C}} \colon X \to \mathrm{Spec}(\mathbb{C})$ induces a pushforward at the level of cohomology $f_* \colon \mathrm{H}^2(X, \mathbb{R}) \to \mathrm{H}^0(\mathrm{Spec}(\mathbb{C}), \mathbb{R})$ given by integration along the fibers. Moreover, there is a natural morphism $a \colon \mathrm{H}^0(\mathrm{Spec}(\mathbb{C}), \mathbb{R}) = \mathrm{H}^{0,0}(\mathrm{Spec}(\mathbb{C})) \to \widehat{\mathrm{CH}}^1(\mathcal{S})$ defined by $a(\mathrm{cl}(\eta)) = [0, [\eta]]$.

A more general definition of the morphism a is given with Theorem III.1.1 in [59]. We can finally state the arithmetic Riemann–Roch theorem, for a proof see [29].

Theorem 1.5.3. *Let $f \colon \mathcal{X} \to \mathcal{S}$ be an arithmetic surface, then we have the following equality of elements of $\widehat{\mathrm{CH}}^1(\mathcal{S})$:*

$$\widehat{\mathrm{c}}_1(\lambda(\overline{\mathcal{L}})_Q) = f_* \left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}}) \, \widehat{\mathrm{td}}(f) \right)^{(1)} + a \left(f_*(-\mathrm{ch}(L) \, \mathrm{td}(T_X) \, \mathrm{R}(T_X))^{(0)} \right). \qquad \square$$

Remark 1.5.4. The term $a\left(f_*(-\mathrm{ch}(L)\,\mathrm{td}(T_X)\,\mathrm{R}(T_X))\right)$, expressing the failure of commutativity of the diagram (1.5.2), is independent of the metric, and is therefore usually called the topological term.

Corollary 1.5.5. *Applying the arithmetic degree to both sides of the arithmetic Riemann–Roch theorem we obtain the following equality of real numbers:*

$$\widehat{\deg}\, \widehat{\mathrm{c}}_1 \left(\lambda(\overline{\mathcal{L}})_Q \right) = \frac{1}{12} \left(6\, \overline{\mathcal{L}}.\overline{\mathcal{L}} - 6\, \overline{\mathcal{L}}.\overline{\omega}_{\mathcal{X}} + \overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}} \right) + \delta_f + \frac{2\zeta'(-1) + \zeta(-1)}{2} \int_X \mathrm{c}_1(\overline{\omega}_X),$$

where δ_f is a real number only dependent on the arithmetic surface $f \colon \mathcal{X} \to \mathcal{S}$.

Proof. To prove that

$$\widehat{\deg}\, a\left(f_*(-\mathrm{ch}(L)\,\mathrm{td}(T_X)\,\mathrm{R}(T_X)^{(0)})\right) = \frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1(\overline{\omega}_X)$$

it is enough to review definitions 1.5.1 and remark 1.5.2, observing that $\mathrm{c}_1(\overline{T}_X) = -\mathrm{c}_1(\overline{\omega}_X)$. On the other hand

$$\delta_f = \widehat{\deg}\, f_*\left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}})\,\widehat{\mathrm{td}}(f)\right)^{(1)} - \widehat{\deg}\, f_*\left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}})\left(1 - \frac{\widehat{\mathrm{c}}_1(\overline{\omega}_{\mathcal{X}})}{2} + \frac{\widehat{\mathrm{c}}_1(\overline{\omega}_{\mathcal{X}})^2}{12}\right)\right)^{(1)}$$

is independent of the metric, because the smooth locus of f does not contribute to it. A more accurate description of the contribution of the singular fibers to δ_f is given in [24, Chap. 6]. Now, the validity of

$$\widehat{\deg}\, f_*\left(\widehat{\mathrm{ch}}(\overline{\mathcal{L}})\left(1 - \frac{\widehat{\mathrm{c}}_1(\overline{\omega}_{\mathcal{X}})}{2} + \frac{\widehat{\mathrm{c}}_1(\overline{\omega}_{\mathcal{X}})^2}{12}\right)\right)^{(1)} = \frac{1}{12}\left(6\,\overline{\mathcal{L}}.\overline{\mathcal{L}} - 6\,\overline{\mathcal{L}}.\overline{\omega}_{\mathcal{X}} + \overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right),$$

is verified using the construction of the arithmetic intersection numbers, definition 1.3.14. □

1.6 Example: $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(m)$ with the Fubini–Study metric

The aim of this example is to explicitly compute all the terms appearing in corollary 1.5.5 for powers of the tautological bundle equipped with the Fubini–Study metric on $\mathbb{P}^1_{\mathbb{Z}}$. Let us consider the arithmetic surface $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ equipped with the natural morphism f to $\mathcal{S}$. On it we consider the tautological bundles $\mathcal{O}(m) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(m) \in \mathrm{Pic}(\mathbb{P}^1_{\mathbb{Z}})$ for any $m \in \mathbb{Z}$. Since $\deg\colon \mathrm{Pic}(\mathbb{P}^1_{\mathbb{Z}}) \to \mathbb{Z}$ is an isomorphism, their classes form a complete set of representatives for the elements of $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{Z}})$. The Fubini–Study metric on $\mathbb{P}^1_{\mathbb{C}}$ is Kähler, and it induces natural hermitian metrics on each $\mathcal{O}(m)$, let us denote them by $\overline{\mathcal{O}}(m)$.
By direct check of the degree: $\omega_{\mathcal{X}} \simeq \mathcal{O}(-2)$, and the isomorphism is an isometry. Thus, by remark 1.4.19, the left hand side is invariant by the transformation $m \mapsto -m-2$.

Lemma 1.6.1. *The L^2-volume of the determinant of cohomology has the value*

$$\begin{aligned}\widehat{\deg}\left(\left(\det \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m)) \otimes \det \mathrm{H}^1(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))^\vee\right)_{L^2,\mathrm{FS}}\right) &= -\frac{1}{2}\sum_{j=0}^{m} \log \frac{(m-j)!\,j!}{(m+1)!} \\ &\quad -\frac{1}{2}\sum_{j=0}^{-m-2} \log \frac{(-m-2-j)!\,j!}{(-m-1)!}.\end{aligned}$$

Proof. The isomorphism

$$\mathrm{H}^1(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(m))^\vee \simeq \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(-2-m))$$

given by Serre's duality induces an isometry for the L^2-metric. Therefore, it is enough to compute

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))_{L^2,\mathrm{FS}}\right)$$

for any $m \in \mathbb{Z}$. On $\mathbb{P}^1_{\mathbb{C}}$ we choose homogeneous coordinates $[x_0 : x_1]$ and let

$$U_0 = \{[x_0 : x_1] \in \mathbb{P}^1_{\mathbb{C}} \,|\, x_1 \neq 0\}$$
$$U_1 = \{[x_0 : x_1] \in \mathbb{P}^1_{\mathbb{C}} \,|\, x_0 \neq 0\},$$

be two affine open subsets covering $\mathbb{P}^1_{\mathbb{C}}$ with coordinates $z_0 = \frac{x_0}{x_1}$ and $z_1 = \frac{x_1}{x_0}$, respectively. For $m < 0$ the only global section of $\mathcal{O}(m)$ is the trivial one. Thus,

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))_{L^2,\mathrm{FS}}\right) = -\log(\mathrm{vol}_{\mathrm{FS}}\mathbb{P}^1_{\mathbb{C}}) = 0,$$

since the Fubini–Study metric, whose volume form on the chart U_0 reads

$$\mu_{\mathrm{FS}}(z_0) = \frac{i}{2\pi}\frac{\mathrm{d}z_0 \wedge \mathrm{d}\bar{z}_0}{(1+|z_0|^2)^2},$$

is chosen in such a way that $\mathrm{vol}_{\mathrm{FS}}\mathbb{P}^1_{\mathbb{C}} = 1$. On the other hand, if $m \geq 0$ a set of generators for $\mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(m))$ as a $\mathbb{Z}$-module is given by the $m+1$ monic homogeneus polynomials of degree m:

$$\{x_0^m, x_0^{m-1}x_1, ..., x_0x_1^{m-1}, x_1^m\}.$$

Then

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))_{L^2,\mathrm{FS}}\right) = -\frac{1}{2}\log\left(\det\left(\langle x_0^{m-j}x_1^j, x_0^{m-h}x_1^h\rangle_{L^2,\mathrm{FS}}\right)_{j,h=0,\dots,m}\right),$$

where the Fubini–Study pointwise inner product on U_0 is given by

$$\langle x_0^{\alpha_0}x_1^{\alpha_1}, x_0^{\beta_0}x_1^{\beta_1}\rangle_{\mathrm{FS}}(z_0) = \frac{z_0^{\alpha_0}\bar{z}_0^{\beta_0}}{(1+|z_0|^2)^{\alpha_1+\alpha_0}},$$

and the Fubini–Study L^2-inner product is given by

$$\langle x_0^{\alpha_0}x_1^{\alpha_1}, x_0^{\beta_0}x_1^{\beta_1}\rangle_{L^2,\mathrm{FS}} = \int_{U_0}\frac{z_0^{\alpha_0}\bar{z}_0^{\beta_0}}{(1+|z_0|^2)^{\alpha_1+\alpha_0}}\mu_{\mathrm{FS}}(z_0).$$

The homogeneus polynomials are pairwise orthogonal for the Fubini–Study metric. Indeed let $\alpha_0 \neq \beta_0$, then,

$$\begin{aligned}\langle x_0^{\alpha_0}x_1^{\alpha_1}, x_0^{\beta_0}x_1^{\beta_1}\rangle_{L^2,\mathrm{FS}} =& \int_{U_0}\frac{z_0^{\alpha_0}\bar{z}_0^{\beta_0}}{(1+|z_0|^2)^{\alpha_1+\alpha_0}}\mu_{\mathrm{FS}}(z_0)\\ =&\frac{1}{\pi}\left(\int_0^{2\pi} e^{i(\alpha_0-\beta_0)\theta}\mathrm{d}\theta\right)\left(\int_0^{\infty}\frac{r^{\alpha_0+\beta_0+1}}{(1+r^2)^{\alpha_1+\alpha_0+2}}\mathrm{d}r\right),\end{aligned}$$

where in the second equality we changed coordinates to $z_0 = re^{i\theta}$. The latter expression is zero because of the assumption $\alpha_0 - \beta_0 \neq 0$. We compute the L^2-norm of these homogeneous polynomials in term of polar coordinates

$$
\begin{aligned}
\|x_0^{\alpha_0}x_1^{\alpha_1}\|^2_{L^2,\mathrm{FS}} =& \frac{i}{2\pi}\int\limits_{U_0} \frac{|z_0|^{2\alpha_0}}{(1+|z_0|^2)^{2+\alpha_1+\alpha_0}}\mathrm{d}z_0\wedge\mathrm{d}\bar{z}_0\\
=& 2\int\limits_0^\infty \frac{r^{2\alpha_0+1}}{(1+r^2)^{2+\alpha_0+\alpha_1}}\mathrm{d}r.
\end{aligned}
$$

Using the formulae

$$
\int\limits_0^\infty \frac{x^s}{(1+x^2)^t}\mathrm{d}x = \frac{s-1}{2(t-1)}\int\limits_0^\infty \frac{x^{s-2}}{(1+x^2)^{t-1}}\mathrm{d}x \qquad (2t-1>s>1)\,,
$$

and

$$
\int\limits_0^\infty \frac{x}{(1+x^2)^n}\mathrm{d}x = \frac{1}{2(n-1)} \qquad (n>1)\,,
$$

we have

$$
\|x_0^{\alpha_0}x_1^{\alpha_1}\|^2_{L^2,\mathrm{FS}} = \frac{\alpha_0!\,\alpha_1!}{(\alpha_0+\alpha_1+1)!}.
$$

Summing over the diagonal completes the proof. □

The combination of formulae (1.4.1) and (1.4.2) implies

$$
\det{}'\left(\Delta^1_{\overline{\mathcal{O}}(m)}\right) = \det{}'\left(\Delta^0_{\overline{\mathcal{O}}(m)}\right) = \det{}'\left(\Delta^0_{\overline{\mathcal{O}}(-2-m)}\right).
$$

Thus, it is enough to compute $\det{}'\left(\Delta^0_{\overline{\mathcal{O}}(m)}\right)$ for $m\geq -1$. We quote the non-zero eigenvalues of $\Delta^0_{\overline{\mathcal{O}}(m)}$ from [37]: they are $\{k(k+m+1)\}$ with multiplicity $2k+m+1$ for any $k\in\mathbb{Z}_{>0}$. Then, the spectral zeta function has the explicit expression

$$
\zeta_{\overline{\mathcal{O}}(m)}(s) = \sum_{k=1}^\infty \frac{2k+m+1}{(k(k+m+1))^s} \qquad (\mathrm{Re}(s)>1)\,.
$$

We apply proposition 3.1 of [65] for the evaluation of the derivative of its analytic continuation in $s=0$. The result reads

$$
\begin{aligned}
\det{}'\left(\Delta^0_{\overline{\mathcal{O}}(m)}\right) =& -\frac{\mathrm{d}}{\mathrm{d}s}\left(\zeta_{\overline{\mathcal{O}}(m)}(s)\right)_{s=0}\\
=& 2\sum_{j=1}^{m+1}(m+1-j)\log(j) + \frac{(m+1)^2}{2} - 4\zeta'(-1) - (m+1)\log((m+1)!).
\end{aligned}
\tag{1.6.1}
$$

To compute the intersection numbers we recall $\overline{\omega}_{\mathcal{X}} \simeq \overline{\mathcal{O}}(-2)$ and we use the bilinearity to obtain

$$\frac{1}{12}\left(6\,\overline{\mathcal{O}}(m).\overline{\mathcal{O}}(m) - 6\,\overline{\mathcal{O}}(m).\overline{\omega}_{\mathcal{X}} + \overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right) = \frac{3m^2+6m+2}{6}\,\overline{\mathcal{O}}(1).\overline{\mathcal{O}}(1).$$

To compute the number $\overline{\mathcal{O}}(1).\overline{\mathcal{O}}(1)$ we apply proposition 1.3.15 with the non-zero global sections x_0 and x_1 of $\mathcal{O}(1)$. Since the divisors of x_0 and x_1 are disjoint at any finite fiber of $\mathbb{P}^1_{\mathbb{Z}}$, the intersection number reduces to

$$\overline{\mathcal{O}}(1).\overline{\mathcal{O}}(1) = -(\log\|x_0\|_{\mathrm{FS}})[\operatorname{div}(x_1)] - \int\limits_{\mathbb{P}^1(\mathbb{C})} \log\|x_0\|_{\mathrm{FS}}\,\mathrm{c}_1(\overline{\mathcal{O}}(1)).$$

The divisor of x_1 is the point $z_1 = 0$ in U_1, where $\|x_0\|_{\mathrm{FS}}$ has the value 1. Thus, the first term is zero. To compute the integral term a short computation shows $\mathrm{c}_1(\overline{\mathcal{O}}(1)) = \mu_{\mathrm{FS}}$. Directly passing to polar coordinates $z_0 = re^{i\theta}$, we compute

$$\begin{aligned}
-\int\limits_{U_0} \log\|x_0\|_{\mathrm{FS}}\,\mathrm{c}_1(\overline{\mathcal{O}}(1)) &= -\int\limits_0^{2\pi}\int\limits_0^{\infty} \frac{1}{2}\log\left(\frac{1}{1+r^2}\right)\frac{i}{2\pi}\frac{1}{(1+r^2)^2}(-2ir)\mathrm{d}r\wedge\mathrm{d}\theta \\
&= -\int\limits_0^{\infty}\log\left(\frac{1}{1+r^2}\right)\frac{r}{(1+r^2)^2}\mathrm{d}r = \frac{1}{2}.
\end{aligned}$$

Summing up

$$\frac{1}{12}\left(6\,\overline{\mathcal{O}}(m).\overline{\mathcal{O}}(m) - 6\,\overline{\mathcal{O}}(m).\overline{\omega}_{\mathcal{X}} + \overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right) = \frac{3m^2+6m+2}{12}.$$

To compute the topological term we use

$$\int\limits_{\mathbb{P}^1_{\mathbb{C}}} \mathrm{c}_1(\overline{\omega}_X) = \deg(\omega_X) = -2,$$

where the first equality is a non-trivial result arising from the Poincaré–Lelong formula and Stokes' theorem. Thus,

$$\frac{2\zeta'(-1)+\zeta(-1)}{2}\int\limits_{\mathbb{P}^1_{\mathbb{C}}} \mathrm{c}_1(\overline{\omega}_X) = -2\zeta'(-1) + \frac{1}{12}.$$

To verify the equality of the two sides of corollary 1.5.5 we assume $m \geq -1$. The left hand side of corollary decomposes as

$$\begin{aligned}
\widehat{\deg}\,\widehat{c}_1\left(\lambda(\overline{\mathcal{O}}(m))_Q\right) =& \widehat{\deg}\left(\det \mathrm{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))_{L^2.\mathrm{FS}}\right) + \widehat{\deg}\left(\det \mathrm{H}^1(\mathbb{P}^1_{\mathbb{Z}}, \overline{\mathcal{O}}(m))^{\vee}_{L^2,\mathrm{FS}}\right) \\
&+ \frac{1}{2}\log(\det{}'\Delta^1_{\overline{\mathcal{O}}(m)}).
\end{aligned}$$

By lemma 1.6.1 and formula (1.6.1),

$$\widehat{\deg}\,\widehat{c}_1\left(\lambda(\overline{\mathcal{O}}(m))_Q\right) = -\frac{1}{2}\sum_{j=0}^{m}\log\frac{(m-j)!\,j!}{(m+1)!} + \sum_{j=1}^{m+1}(m+1-j)\log(j) + \frac{(m+1)^2}{4} - 2\zeta'(-1) - \frac{(m+1)\log((m+1)!)}{2}.$$

Finally using the relation $\sum\limits_{j=0}^{m}\log\left((m-j)!j!\right) = 2\sum\limits_{j=1}^{m+1}(m+1-j)\log(j)$,

$$\widehat{\deg}\,\widehat{c}_1\left(\lambda(\overline{\mathcal{O}}(m))_Q\right) = \frac{(m+1)^2}{4} - 2\zeta'(-1).$$

As predicted by corollary 1.5.5, the sum of the intersection numbers with the topological term yields the same result. Moreover, as predicted by remark 1.4.19, the two sides of the formula are invariant by the transformation $m \mapsto -m-2$.

Chapter 2

The regularized determinant of a generalized Laplacian

In this chapter we extend the regularization approach of Jorgenson and Lundelius [39] to the determinant of the Laplacian on cusp forms of higher weight on a complex non-compact modular curve. Afterwards, proceeding along the lines of computations done in the compact case, most notably by D'Hoker and Phong [17] and by Sarnak [55], we provide an explicit expression for it.

2.1 The line bundle of cusp forms on a hyperbolic Riemann surface

On the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ we consider the hyperbolic metric

$$\mathrm{d}s^2_{\mathrm{hyp}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2},$$

whose volume form is

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{y^2} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{y^2},$$

and which has associated distance function

$$d_{\mathrm{hyp}}(z, w) = \operatorname{arccosh}\left(1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\right). \tag{2.1.1}$$

Let Γ be a cofinite torsion-free and discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. By abuse of notation we write elements in Γ as matrices, in particular we write

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

for a generic element $\gamma \in \Gamma$. The group Γ acts by fractional linear transformations on $\mathbb{H}$:

$$\gamma(z) := \frac{az+b}{cz+d} \qquad (\gamma \in \Gamma,\ z \in \mathbb{H}).$$

We consider the Riemann surface $Y(\Gamma) = \Gamma\backslash\mathbb{H}$, and its compactification $X(\Gamma) = \overline{\Gamma\backslash\mathbb{H}}$ by adding the so-called cusps. We let g be the genus of $X(\Gamma)$ and denote the cusps by $P_1, ..., P_p$. Moreover, we fix a fixed fundamental domain $\mathcal{F}_\Gamma$ for the action of Γ on $\mathbb{H}$.

The hyperbolic metric on $\mathbb{H}$ descends to the quotient, yielding a smooth hermitian metric on $Y(\Gamma)$ which is singular at the cusps when extended to $X(\Gamma)$. In particular, for each cusp there exists a local coordinate q centered at the cusp such that the hyperbolic squared line element is given by the expression

$$\mathrm{d}s^2_{\mathrm{hyp}} = \frac{|\mathrm{d}q|^2}{(|q| \log |q|)^2}. \tag{2.1.2}$$

A local coordinate for the cusp at infinity, expressed in terms of $z \in \mathbb{H}$, is $q = e^{2\pi i z}$.

Notation 2.1.1. Let G be a subgroup of Γ, in the sequel G will denote either Γ or the subgroup $\langle\gamma\rangle$ generated by $\gamma \in \Gamma$. In general we will not distinguish between points on $\mathbb{H}$ or on the quotient $G\backslash\mathbb{H}$. Specifically, we always fix a fundamental domain $\mathcal{F}_\Gamma \subseteq \mathcal{F}_G \subseteq \mathbb{H}$ for the quotient of $\mathbb{H}$ by the group G. Then, a point $z \in G\backslash\mathbb{H}$ also equivalently denotes its preimage in $\mathcal{F}_G$, and a point $z \in \mathbb{H}$ also equivalently denotes its image in $G\backslash\mathbb{H}$. Similarly we do not make distinctions between differential forms on $G\backslash\mathbb{H}$ and G-invariant differential forms on $\mathbb{H}$.
Moreover, since the group Γ is fixed, by abuse of notation we write from now on

$$X = X(\Gamma),$$

as in the previous chapter, the symbol X denotes a compact Riemann surface.

We now define the line bundle of modular forms M_k.

Definition 2.1.2. Let $k \in \mathbb{Z}$, a *modular form of weight* $2k$ *for* Γ is a holomorphic function $f\colon \mathbb{H}\to\mathbb{C}$ that fulfills the functional equation

$$f(\gamma(z)) = (cz+d)^{2k} f(z) \qquad (\gamma \in \Gamma,\, z \in \mathbb{H}),$$

and that is holomorphic at all the cusps.

Remark 2.1.3. The factor $(cz+d)^{2k}$ is a 1-cocycle for Γ and defines a line bundle on $Y(\Gamma)$ which extends to the compactification X. We denote this line bundle by M_k and call it the *line bundle of modular forms of weight* $2k$ *for* Γ.

Let the cusp divisor D on X be defined by the formula

$$D = \sum_{j=1}^{p} P_j.$$

Using its associated line bundle we define the line bundle of cusp forms S_k.

Definition 2.1.4. We define the *line bundle of cusp form of weight* $2k$ *for* Γ by the relation

$$S_k = M_k \otimes \mathcal{O}_X(D)^\vee, \tag{2.1.3}$$

Holomorphic global sections of S_k are cusp forms of weight $2k$ for Γ, i.e., modular forms of weight $2k$ for Γ that vanish at the cusps.

Definition 2.1.5. The *Petersson metric on* M_k is given by the formula

$$\langle f(z), g(z)\rangle_{\overline{M}_k,\mathrm{Pet}} := f(z)\overline{g(z)}\, y^{2k} \qquad \big(f, g \in A^{0,0}(X, M_k),\ z \in \mathbb{H}\big)\,.$$

Let $\overline{\mathcal{O}}_X(D)$ be the line bundle associated to the cusp divisor D equipped with the trivial metric. By abuse of notation, the metric on S_k that makes equality (2.1.3) an isometry is called the *Petersson metric on* S_k, and it is also given by the formula

$$\langle f(z), g(z)\rangle_{\overline{S}_k,\mathrm{Pet}} := f(z)\overline{g(z)}\, y^{2k} \qquad \big(f, g \in A^{0,0}(X, S_k),\ z \in \mathbb{H}\big)\,.$$

We denote by $\overline{M}_k = \Big(M_k, \|\cdot\|_{\overline{M}_k,\mathrm{Pet}}\Big)$ and $\overline{S}_k = \Big(S_k, \|\cdot\|_{\overline{S}_k,\mathrm{Pet}}\Big)$ the resulting hermitian line bundles.
The hyperbolic metric naturally induces a singular hermitian metric on the holomorphic cotangent bundle ω_X of X. Let $\overline{\omega}_X$ be the resulting singular hermitian line bundle. For a holomorphic coordinate z, the norm of the section $\mathrm{d}z$ of ω_X is given by

$$\|\mathrm{d}z\|_{\mathrm{hyp}} = y \qquad (z \in \mathbb{H})\,. \tag{2.1.4}$$

Observation 2.1.6. For $k \in \mathbb{Z}$, we have the isometry

$$\overline{M}_k \simeq \big(\overline{\omega}_X \otimes \overline{\mathcal{O}}_X(D)\big)^{\otimes k}\,.$$

Proof. The claim follows from the relation

$$\mathrm{d}(\gamma(z)) = \frac{\mathrm{d}z}{(cz+d)^2} \qquad (\gamma \in \Gamma,\ z \in \mathbb{H})\,,$$

and from equation (2.1.4). □

The Petersson metrics on M_k and S_k are both singular at the cusps, but with different behaviors. Let U be a neighborhood of the cusp P_j which trivializes both M_k and S_k, and let us consider local sections $f \in \mathrm{H}^0(U, M_k)$ and $g \in \mathrm{H}^0(U, S_k)$ such that they have both order 0 at the cusp P_j. Then

$$\|f(q)\|_{\overline{M}_k,\mathrm{Pet}} = \frac{1}{(2\pi)^k} \cdot |f(q)| \cdot (-\log|q|)^k\,,$$

and

$$\|g(q)\|_{\overline{S}_k,\mathrm{Pet}} = \frac{|q|}{(2\pi)^k} \cdot \left|\frac{g(q)}{q}\right| \cdot (-\log|q|)^k\,.$$

Remark 2.1.7. The last expressions show that $\overline{M}_k$ is a logarithmically singular hermitian line bundle, in the sense of Kühn [41, definition 3.1], but $\overline{S}_k$ is not. In [31, definition 4.2.4] Hahn defines the class of fairly good hermitian line bundles, to which $\overline{S}_k$ belongs.

The Petersson norm induces an inner product on the space of global section of S_k, which, by abuse of notation, we denote by the same symbol.

Definition 2.1.8. Let $f, g \in A^{0,0}(X, S_k)$, their *Petersson inner product* is defined by the formula

$$\langle f, g \rangle_{L^2, \overline{S}_k, \mathrm{Pet}, \mathrm{hyp}} := \int_X \langle f(z), g(z) \rangle_{\overline{S}_k, \mathrm{Pet}}\, \mu_{\mathrm{hyp}}(z) = \int_{\mathcal{F}_\Gamma} f(z) \overline{g(z)}\, y^{2k} \mu_{\mathrm{hyp}}(z).$$

Let us observe that the Petersson inner product is the L^2-product of definition 1.4.5 obtained from the Petersson norm $\|\cdot\|_{\overline{S}_k, \mathrm{Pet}}$ on S_k and the hyperbolic metric on X. Furthermore, observe that the same expression does not define an inner product for global sections of M_k, because the integral is not always convergent. Summing up, there is not a preferential choice in deciding to work with $\overline{M}_k$ or $\overline{S}_k$. In this work we aim at proving an arithmetic Riemann–Roch theorem for $\overline{S}_k$, but we also point out that Freixas [24, 25] and Freixas–von Pippich [26] work with $\overline{M}_k$.

Notation 2.1.9. The ease notation we drop some subscripts in the expressions of the considered metrics, we thus write

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\overline{M}_k} &:= \langle \cdot, \cdot \rangle_{\overline{M}_k, \mathrm{Pet}}, \\ \langle \cdot, \cdot \rangle_{\overline{S}_k} &:= \langle \cdot, \cdot \rangle_{\overline{S}_k, \mathrm{Pet}}, \\ \langle \cdot, \cdot \rangle_{\mathrm{Pet}} &:= \langle \cdot, \cdot \rangle_{L^2, S_k, \mathrm{Pet}, \mathrm{hyp}}. \end{aligned}$$

In the present singular setting, the Laplacian $\Delta^1_{\overline{S}_{k+1}}$ can be still introduced as in definition 1.4.7 but the definition of its L^2-determinant given in definition 1.4.15 is in general not applicable anymore. The goal of this section is to define and compute the regularized determinant $\det^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)$ for any $k \geq 0$. For technical reasons it is easier for us to work with $\Delta^0_{\overline{M}_{-k}}$ instead, this is justified by equation (1.4.1).

2.2 The hyperbolic heat kernel of weight k

Let $k \geq 0$, in this section we introduce and study the first properties of the heat kernel associated to the Laplacian

$$\Delta_k := 4\, \Delta^0_{\overline{M}_{-k}}, \tag{2.2.1}$$

which we call hyperbolic heat kernel of weight k. Even though nor the hyperbolic metric on X nor the Petersson metric on M_{-k} are smooth, the machinery leading to the definition of the Laplacian Δ_k is still valid. Specifically, the metrics on X and on M_{-k} induce L^2-metrics on $A^{0,0}(X, M_{-k})$ and $A^{0,1}(X, M_{-k})$. The Cauchy–Riemann operator on global sections of M_{-k}

$$\bar{\partial}_{\overline{M}_{-k}} = \partial_{\bar{z}}\, \mathrm{d}\bar{z} :\ A^{0,0}(X, M_{-k}) \longrightarrow A^{0,1}(X, M_{-k})$$

has an adjoint with respect to the L^2-metrics given by

$$\bar{\partial}^*_{\overline{M}_{-k}} = -y^{2(k+1)} \partial_z\, y^{-2k}\, \mathrm{d}z :\ A^{0,1}(X, M_{-k}) \longrightarrow A^{1,1}(X, M_{-k}) \simeq A^{0,0}(X, M_{-k}),$$

where the last isomorphism is given by Hodge duality. Composing the two operators we obtain the desired Laplacian

$$\begin{aligned}\Delta_k &= 4\,\bar\partial^*_{\overline{M}_{-k}}\,\bar\partial_{\overline{M}_{-k}}\\ &= -4y^2\,\partial_z\partial_{\bar z} - 4ik\,y\,\partial_{\bar z}\\ &= -y^2\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) - 2ik\,y\left(\frac{\mathrm{d}}{\mathrm{d}x}+i\frac{\mathrm{d}}{\mathrm{d}y}\right).\end{aligned}$$

To obtain an explicit expression for the heat kernel associated to Δ_k we show that it is related to an operator with known heat kernel. For each $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$ we write

$$j(\gamma, z) := \frac{cz+d}{c\bar z + d}.$$

We observe that $j(\gamma, z)$ satisfies the 1-cocycle relation

$$j(\gamma\delta, z) = j(\gamma, \delta(z))\,j(\delta, z) \qquad (\gamma, \delta \in \Gamma,\ z \in \mathbb{H})\,, \tag{2.2.2}$$

which is derived by direct computation or using formula (1.2.4) of [57]. The 1-cocycle factor $j(\gamma, z)^{-k}$ induces the complex line bundle A_{-k} on X, which becomes the hermitian line bundle $\overline{A}_{-k} = (A_{-k}, |\cdot|)$ once equipped with the trivial metric. The multiplication by y^k induces an isometry of hermitian line bundles between $\overline{A}_{-k}$ and $\overline{M}_{-k}$, i.e., we have

$$\|g(z)\|_{\overline{A}_{-k}} = |g(z)| = \|g(z)y^k\|_{\overline{M}_{-k}} \qquad \big(g \in A^{0,0}(X, A_{-k}),\ z \in \mathbb{H}\big)\,.$$

Now, we consider the hyperbolic Laplacian

$$D_k = -4y^2\,\partial_z\partial_{\bar z} - 2ik\,y\,(\partial_z + \partial_{\bar z}).$$

The Laplacian D_k acts on the space $A^{0,0}(X, A_{-k})$ of smooth sections of A_{-k}, namely functions $g\colon \mathbb{H}{\to}\mathbb{C}$ satisfying the functional equation

$$g(\gamma(z)) = \left(\frac{c\bar z + d}{cz+d}\right)^k g(z) \qquad (\gamma \in \Gamma,\ z \in \mathbb{H})\,.$$

A straightforward computation gives the relation between Δ_k and the conjugate of D_k under the isometry induced by the multiplication for y^k:

$$\Delta_k(f(z)) = y^k D_k(y^{-k}\,f(z)) + k(k+1)\,f(z) \qquad \big(f \in A^{0,0}(X, M_{-k}),\ z \in \mathbb{H}\big)\,. \tag{2.2.3}$$

Combining formula (2.2.3) with the fact that the spectral radius of D_k is $-k(k-1)$, as stated on page 11 of [23], we obtain that the discrete eigenvalues of Δ_k are bounded from below by $2k \geq 0$.

Let $K_{\Delta_k}(t; z, w)$ and $K_{D_k}(t; z, w)$ be the heat kernels on $\mathbb{H}$ associated to the Laplacians Δ_k and D_k, respectively, according to definition 1.4.10. Using formula (2.2.3) they can be related.

Lemma 2.2.1. *For each $t > 0$, and $z, w \in \mathbb{H}$,*

$$K_{\Delta_k}(t; z, w) = e^{-tk(k+1)} \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} K_{D_k}(t; z, w).$$

Proof. We verify that the right hand side of the claim satisfies definition 1.4.10 assuming that $K_{D_k}(t; z, w)$ does. The regularity is trivial. To verify that it is a solution of the heat equation we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-tk(k+1)} \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} K_{D_k}(t; z, w)\right) &= -e^{-tk(k+1)} \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} \left(D_{k,z} + k(k+1)\right)\left(K_{D_k}(t; z, w)\right) \\ &= -\Delta_{k,z}\left(e^{-tk(k+1)} \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} K_{D_k}(t; z, w)\right).\end{aligned}$$

To verify that it approximates the δ-distribution for t close to zero we fix an element $f \in A^{0,0}(X, M_{-k})$ with compact support, then

$$\begin{aligned}&\lim_{t\to 0} \int_{\mathbb{H}} e^{-tk(k+1)} \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} K_{D_k}(t; z, w)\, f(w)\, \mu_{\mathrm{hyp}}(w) \\ &\qquad = \mathrm{Im}(z)^k \lim_{t\to 0} \int_{\mathbb{H}} K_{D_k}(t; z, w)\, \frac{f(w)}{\mathrm{Im}(w)^k}\, \mu_{\mathrm{hyp}}(w).\end{aligned}$$

By the properties of $K_{D_k}(t; z, w)$, the last expression equals $f(z)$. □

Notation 2.2.2. We define an auxiliary heat kernel function by the relation

$$K_k(t; z, w) = e^{-tk(k+1)} K_{D_k}(t; z, w).$$

Since we will see, corollary 2.2.4, that the expression just defined is only dependent on the hyperbolic distance of the two space variables, we will use the notation

$$K_k(t; d_{\mathrm{hyp}}(z, w)) = K_k(t; z, w).$$

For each subgroup $G \subseteq \Gamma$, the heat kernel $K_k^G(t; z, w)$ on $G\backslash\mathbb{H}$ associated to Δ_k can be expressed as a Poincaré sum on the heat kernel on $\mathbb{H}$ via the formula

$$K_k^G(t; z, w) = \sum_{\gamma\in G} (cz + d)^{2k} \left(\frac{w - \gamma(\bar{z})}{\gamma(z) - \bar{w}}\right)^{-k} K_{\Delta_k}(t; \gamma(z), w).$$

Using lemma 2.2.1 and the equality

$$(cz + d)^{2k} = \left(\frac{cz + d}{c\bar{z} + d}\right)^k \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(\gamma(z))^k},$$

we obtain

$$K_k^G(t; z, w) = \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} \sum_{\gamma\in G} \left(\frac{cz + d}{c\bar{z} + d}\right)^k \left(\frac{\gamma(z) - \bar{w}}{w - \gamma(\bar{z})}\right)^k K_k(t; \gamma(z), w). \tag{2.2.4}$$

An explicit expression for the heat kernel $K_{D_k}(t; z, w)$ is known, this gives an explicit expression for $K_k(t; z, w)$.

Theorem 2.2.3. *Let $d = d_{\mathrm{hyp}}(z,w)$ be the hyperbolic distance of the space variables, then*

$$K_k(t;z,w) = \frac{\sqrt{2}e^{-t\left(k+\frac{1}{2}\right)^2}}{(4\pi t)^{\frac{3}{2}}} \int_d^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{\cosh(u)-\cosh(d)}} T_{2k}\left(\frac{\cosh\left(\frac{u}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}u \tag{2.2.5}$$

$$= \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t(2k-j)(j+1)} P_{k-j,k}(\cosh(d)) \tag{2.2.6}$$

$$+ \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \int_0^\infty r\tanh(\pi r) e^{-tr^2} P_{\frac{1}{2}+ir,k}(\cosh(d))\,\mathrm{d}r,$$

where the modified Legendre function is

$$P_{s,k}(u) = \left(\frac{2}{1+u}\right)^s {}_2F_1\left(s-k, s+k; 1; \frac{u-1}{u+1}\right). \tag{2.2.7}$$

Credits. The result has been proven, with minor later adjustments, by Fay [23]. The predecessor of formula (2.2.5) is equation (42) in [23]. In [17] D'Hoker and Phong realized that the complicated factor appearing in the integrand of Fay's formula can be rewritten as a Chebyshev polynomial, but in this equation the contribution of the discrete spectrum has been computed twice. This has been pointed out by Fay himself to D'Hoker and Phong [18, page 1004]. The first correct version of this formula is therefore equation (5.16) in [18]:

$$K_{D_k}(t;z,w) = \frac{\sqrt{2}e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}} \int_d^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sqrt{\cosh(u)-\cosh(d)}} T_{-2k}\left(\frac{\cosh\left(\frac{u}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}u,$$

from which formula (2.2.5) is deduced by exploiting the symmetry $T_{-2k}(Z) = T_{2k}(Z)$ and multiplying by $e^{-tk(k+1)}$. The predecessor of formula (2.2.6) is equation (35), again in [23]. Its expression has been simplified by Oshima in [49] as equation (2.1):

$$K_{D_k}(t;z,w) = \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{t(k-j)(k-j-1)} P_{k-j,-k}(\cosh(d))$$
$$+ \frac{1}{8\pi i} \int_{\mathrm{Re}(s)=\frac{1}{2}} \frac{e^{ts(s-1)}\sin(2\pi s)}{\sin(\pi(s+k))\sin(\pi(s-k))} \left(s-\frac{1}{2}\right) P_{s,-k}(\cosh(d))\mathrm{d}s.$$

Formula (2.2.6) follows from the latter equation by means of the change of variables $s = \frac{1}{2} + ir$, the equality

$$\frac{\sin\left(2\pi\left(\frac{1}{2}+ir\right)\right)}{\sin\left(\pi\left(\frac{1}{2}+ir+k\right)\right)\sin\left(\pi\left(\frac{1}{2}+ir-k\right)\right)} = -2i\tanh(\pi r),$$

the relation $P_{s,-k}(\cosh(d)) = P_{s,k}(\cosh(d))$, which is a consequence of the symmetry relation, formula 15.1.1 of [1],

$${}_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z),$$

and finally by multiplying for $e^{-tk(k+1)}$. □

Corollary 2.2.4. *The heat kernel $K_k(t;z,w)$ depends only on t and on the hyperbolic distance of the space variables, and it is monotone decreasing for increasing distances. Moreover, let $G \subseteq \Gamma$ be a subgroup, then*

$$K_k^G(t;z,z) \in \mathbb{R} \qquad (t>0,\, z \in G\backslash\mathbb{H})\,.$$

Proof. The first statement is evident by formula (2.2.5). Regarding the second statement we follow an argument of Hejhal [35, chapter IV, proposition 2.1.1]. A direct computation shows that, for any $\gamma \in G \subseteq \Gamma$ and $z, w \in \mathbb{H}$, we have the equality

$$\frac{\gamma(w)-\gamma(\bar{z})}{\gamma(z)-\gamma(\bar{w})} = \frac{j(\gamma,z)}{j(\gamma,w)}\,\frac{w-\bar{z}}{z-\bar{w}}. \tag{2.2.8}$$

Combining it with the relation $j(\gamma,\gamma^{-1}(w)) = j(\gamma^{-1},w)^{-1}$, from formula (2.2.2), we verify

$$j(\gamma,z)^{-1}\,\frac{w-\gamma(\bar{z})}{\gamma(z)-\bar{w}} = j(\gamma^{-1},w)\,\frac{\gamma^{-1}(w)-\bar{z}}{z-\gamma^{-1}(\bar{w})}.$$

From formula (2.2.4), using the last relation and the fact that $K_k(t;d_{\mathrm{hyp}}(w,\gamma(z))) = K_k(t;d_{\mathrm{hyp}}(z,\gamma^{-1}(w)))$ is real valued, we compute

$$\begin{aligned}
\overline{K_k^G(t;z,w)} &= \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k}\sum_{\gamma\in G} j(\gamma,z)^{-k}\left(\frac{w-\gamma(\bar{z})}{\gamma(z)-\bar{w}}\right)^k K_k(t;\gamma(z),w)\\
&= \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k}\sum_{\gamma\in G} j(\gamma^{-1},w)^{k}\left(\frac{\gamma^{-1}(w)-\bar{z}}{z-\gamma^{-1}(\bar{w})}\right)^k K_k(t;\gamma^{-1}(w),z)\\
&= \frac{\mathrm{Im}(z)^{2k}}{\mathrm{Im}(w)^{2k}} K_k^G(t;w,z).
\end{aligned}$$

This completes the proof of the corollary. $\square$

2.3 A bound on the hyperbolic heat kernel of weight k

In this section we present a bound on $K_k(t;d)$. The proof is unnecessarily strong, but we state it for later reference. To ease the exposition we introduce some additional notation.

Notation 2.3.1. Let f, g be real valued functions defined on some domain, depending on a parameter a. We write $f \ll_a g$ if there exists a real positive constant c_a only dependent on a such that $f \leq c_a \cdot g$ on the considered domain. Moreover, we write $f \approx_a g$ if $f \ll_a g$ and $g \ll_a f$.

Lemma 2.3.2. *For each $k \geq 0$ we consider the auxiliary functions*

$$A_k^{(0,1)}(t;d) = \frac{d\, e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}}\int_0^1 \frac{e^{-\frac{(v+d)^2}{4t}}}{\sqrt{v}}\mathrm{d}v, \tag{2.3.1}$$

$$A_k^{(1,\infty)}(t;d) = \frac{e^{-(2t+d)k}}{t^{\frac{3}{2}}}\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}}\mathrm{d}v. \tag{2.3.2}$$

Then, under the assumption $d \geq \delta > 0$, we have

$$K_k(t;d) \approx_{\delta,k} A_k^{(0,1)}(t;d) + A_k^{(1,\infty)}(t;d).$$

Proof. We start from the expression given in formula (2.2.5). Removing the constants and changing variables to $v = u - d$ we find

$$K_k(t;d) \approx \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{t^{\frac{3}{2}}} \int_0^\infty \frac{(v+d)e^{-\frac{(v+d)^2}{4t}}}{\sqrt{\cosh(v+d)-\cosh(d)}} T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}v.$$

By direct computation, $\cosh(v+d)-\cosh(d) = \frac{e^d(e^v-1)(1-e^{-2d-v})}{2}$. Moreover, we split the integral at $v=1$ to obtain

$$\begin{aligned} K_k(t;d) \approx{}& \frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_0^1 \frac{(v+d)e^{-\frac{(v+d)^2}{4t}}}{\sqrt{e^v-1}\sqrt{1-e^{-2d-v}}} T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}v \\ &+ \frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_1^\infty \frac{(v+d)e^{-\frac{(v+d)^2}{4t}}}{\sqrt{e^v-1}\sqrt{1-e^{-2d-v}}} T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}v. \end{aligned}$$

Since $d \geq \delta > 0$, we have $1-e^{-v-2d} \approx_\delta 1$. Regarding the first expression: $4v \geq e^v - 1 \geq v$, so $(e^v-1)^{-\frac{1}{2}} \approx \frac{1}{\sqrt{v}}$, and $v + d \approx_\delta d$. While $e^{-\frac{v}{2}} \leq (e^v-1)^{-\frac{1}{2}} \leq 2e^{-\frac{v}{2}}$ for $v \geq 1$, therefore substituting in the second term yields

$$\begin{aligned} K_k(t;d) \approx_\delta{}& \frac{d\, e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_0^1 \frac{e^{-\frac{(v+d)^2}{4t}}}{\sqrt{v}} T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}v \\ &+ \frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_1^\infty (v+d)e^{-\frac{(v+d)^2}{4t}-\frac{v}{2}} T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right)}\right) \mathrm{d}v. \end{aligned}$$

Regarding the Chebyshev polynomial it holds $T_{2k}(Z) \approx_k Z^{2k}$ for any $Z \geq 1$. Moreover, we have

$$\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right))} \approx e^{\frac{v}{2}}.$$

Thus,we deduce

$$T_{2k}\left(\frac{\cosh\left(\frac{v+d}{2}\right)}{\cosh\left(\frac{d}{2}\right))}\right) \approx_k e^{vk}.$$

Specifically, if $v \in [0,1]$, the Chebyshev polynomial is bounded by a constant. It follows that

$$K_k(t;d) \approx_{\delta,k} \frac{d\, e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_0^1 \frac{e^{-\frac{(v+d)^2}{4t}}}{\sqrt{v}} \mathrm{d}v + \frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}}}{t^{\frac{3}{2}}} \int_1^\infty (v+d)e^{-\frac{(v+d)^2}{4t}+v\left(k-\frac{1}{2}\right)} \mathrm{d}v.$$

Finally, in the second term we complete squares on the exponential

$$-t\left(k+\frac{1}{2}\right)^2-\frac{d}{2}-\frac{(v+d)^2}{4t}+v\left(k-\frac{1}{2}\right)=-\frac{(v+d-t(2k-1))^2}{4t}-(2t+d)k.$$

This completes the proof of the lemma. □

Lemma 2.3.3. *The auxiliary functions appearing in lemma* 2.3.2 *admit the following bounds:*

$$A_k^{(0,1)}(t;d) \ll \frac{d\,e^{-tk(k+1)-\frac{(t+d)^2}{4t}}}{t^{\frac{3}{2}}}. \tag{2.3.3}$$

If $k=0$,

$$A_0^{(1,\infty)}(t;d) \ll \frac{e^{-\frac{(t+d+1)^2}{4t}}}{\sqrt{t}}. \tag{2.3.4}$$

If $k \geq 1$ *and* $t \in \mathbb{R}_{>0}$,

$$A_k^{(1,\infty)}(t;d) \ll_k \left(1+\frac{1}{\sqrt{t}}\right)e^{-(2t+d)k}. \tag{2.3.5}$$

If $k \geq 1$ *and* $t < \frac{d+1}{2k-1}$,

$$A_k^{(1,\infty)}(t;d) \ll_k \left(1+\frac{1}{\sqrt{t}}\right)e^{-(2t+d)k-\frac{(d-t(2k-1)+1)^2}{4t}}. \tag{2.3.6}$$

Proof. Since the integral $\int_0^1 \frac{dv}{\sqrt{v}}$ has value $\frac{1}{2}$, and the exponential can be trivially estimated by its value at $v=0$, the first bound follows by completing the squares. For the remaining ones we want to estimate the expression

$$A_k^{(1,\infty)}(t;d)=\frac{e^{-(2t+d)k}}{t^{\frac{3}{2}}}\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}}\mathrm{d}v.$$

We examine the integral separately and compute

$$\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}}\mathrm{d}v = 4t\int_1^\infty \left(\frac{v+d-t(2k-1)}{2\sqrt{t}}\right)e^{-\frac{(v+d-t(2k-1))^2}{4t}}\frac{\mathrm{d}v}{2\sqrt{t}}$$
$$+\,2t^{\frac{3}{2}}(2k-1)\int_1^\infty e^{-\frac{(v+d-t(2k-1))^2}{4t}}\frac{\mathrm{d}v}{2\sqrt{t}}.$$

Using $\frac{v+d-t(2k-1)}{2\sqrt{t}}$ as a variable, we integrate the first term and we express the second one as a complementary error functio to find

$$\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}}\mathrm{d}v = 2t\,e^{-\frac{(d-t(2k-1)+1)^2}{4t}}+t^{\frac{3}{2}}(2k-1)\sqrt{\pi}\ \mathrm{erfc}\left(\frac{d-t(2k-1)+1}{2\sqrt{t}}\right).$$

We separate the cases $k=0$ and $k\geq 1$. Let $k=0$, then

$$\int_1^\infty (v+d)e^{-\frac{(t+d+1)^2}{4t}}\,\mathrm{d}v = 2t\,e^{-\left(\frac{t+d+1}{2\sqrt{t}}\right)^2} - t^{\frac{3}{2}}\sqrt{\pi}\,\mathrm{erfc}\left(\frac{t+d+1}{2\sqrt{t}}\right) \ll t\,e^{-\left(\frac{t+d+1}{2\sqrt{t}}\right)^2}.$$

Where we estimated the difference by its positive summand. The desired estimate (2.3.4) follows. Now we consider $k\geq 1$, in this case we diversify the bound depending on t. For any t we have the trivial bound

$$\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}} \ll_k t+t^{\frac{3}{2}},$$

which implies (2.3.5). But for t small, i.e., if $t<\frac{d+1}{2k-1}$, it is possible to provide a better estimate. Since $\frac{d-t(2k-1)+1}{2\sqrt{t}}>0$, the bound $\mathrm{erfc}(x)\leq e^{-x^2}$, as in formula (5) of [15], gives $\mathrm{erfc}\left(\frac{d-t(2k-1)+1}{2\sqrt{t}}\right)\leq e^{-\frac{(d-t(2k-1)+1)^2}{4t}}$. Therefore

$$\int_1^\infty (v+d)e^{-\frac{(v+d-t(2k-1))^2}{4t}} \ll_k \left(t+t^{\frac{3}{2}}\right)\, e^{-\frac{(d-t(2k-1)+1)^2}{4t}},$$

which in turn implies (2.3.6). $\square$

We can now state the claimed bound for the heat kernel.

Proposition 2.3.4. *We fix $\delta>0$. Then, for any $d\geq\delta$,*

$$K_k(t;d)\ll_{\delta,k,t} e^{-\frac{d^2}{4t}}.$$

Proof. By lemma 2.3.2, to prove the proposition it is enough to show that both $A_k^{(0,1)}(t;d)$ and $A_k^{(1,\infty)}(t;d)$ can be bounded by $c(t)e^{-\frac{d^2}{4t}}$, where $c(t)$ is a constant only depending on t. To do so, we proceed from the estimates of lemma 2.3.3. Regarding the term $A_k^{(0,1)}(t;d)$, we estimate its bound (2.3.3). By expanding the square of the exponential we obtain

$$\frac{d\,e^{-tk(k+1)-\frac{(t+d)^2}{4t}}}{t^{\frac{3}{2}}} = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{t^{\frac{3}{2}}}de^{-\frac{d}{2t}}e^{-\frac{d^2}{4t}}.$$

Since $\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{t^{\frac{3}{2}}}\approx_t 1$ and $de^{-\frac{d}{2t}}\ll_t 1$, this term satisfies the claimed bound. Also regarding the term $A_0^{(1,\infty)}(t;d)$, expanding the square of the exponential in (2.3.4) shows the same bound. To estimate the term $A_k^{(1,\infty)}(t;d)$ for $k\geq 1$ we bound the estimate occurring in (2.3.5) under the assumption $t\geq\frac{d+1}{2k-1}$, since, if this is not the case, we have the stronger estimate (2.3.6) that we examine next. The bound $t\geq\frac{d+1}{2k-1}$ implies

$$e^{-(2t+d)k}\leq e^{-\left(\frac{2(d+1)}{2k-1}+d\right)k} = e^{-(d(2k+1)+2)\frac{k}{2k-1}}\leq e^{-d\left(k+\frac{1}{2}\right)}.$$

Now, the estimate

$$e^{-d\left(k+\frac{1}{2}\right)} \leq e^{-\frac{d^2}{4t}}$$

is implied by $k + \frac{1}{2} \geq \frac{d}{4t}$, which follows from the hypothesis $2k - 1 \geq \frac{d+1}{t}$. It remains to examine the term $A_k^{(1,\infty)}(t; d)$ with $k \geq 1$ and $t < \frac{d+1}{2k-1}$, we proceed from its estimate in formula (2.3.6). Expanding the square in the exponential we obtain

$$e^{-(2t+d)k-\frac{(d-t(2k-1)+1)^2}{4t}} = e^{-(2t+d)k-\frac{d^2}{4t}+\frac{2d(2k-1)}{4}-\frac{t(2k-1)^2}{4}}.$$

Observing that $-(2t+d)k + \frac{2d(2k-1)}{4} \leq 0$ for any t, d and k, completes the estimate of this last term. □

2.4 The hyperbolic regularization of the trace of the heat kernel

The crucial obstruction in mimicking the construction of the determinant of the Laplacian as in the compact case, definition 1.4.15, is that the heat kernel $K_k^\Gamma(t; z, w)$ is not trace class. Namely the expression

$$\int_X K_k^\Gamma(t; z, z)\, \mu_{\text{hyp}}(z)$$

occurring in definition 1.4.14 is in general not convergent. To overcome this difficulty we extend the regularization approach of Jorgenson and Lundelius, section 1 of [39], from $k = 0$ to $k \geq 0$.

An important tool for the regularization is the decomposition of Γ, and of its subgroups, as in the derivation of the Selberg trace formula. Let $G \subseteq \Gamma$ be a subgroup of Γ, then we fix maximal sets $H(G)$ and $P(G)$ of inconjugate primitive hyperbolic and parabolic elements, respectively, such that if γ occurs then also γ^{-1} does. Here we assume the convention that the identity is not a parabolic element. Then, denoting by G_γ the centralizer of γ, and following a classic argument, see for example McKean [45, page 230], the following disjoint decomposition holds

$$G = \{\text{id}\} \cup \bigcup_{\gamma \in H(G) \cup P(G)} \bigcup_{[\eta] \in G_\gamma \backslash G} \bigcup_{n=1}^{\infty} \eta^{-1}\gamma^n\eta. \tag{2.4.1}$$

Let us remark that elements in $H(G)$ and $P(G)$, are not necessarily powers of elements in $H(\Gamma)$ and $P(\Gamma)$, respectively. Indeed there could be distinct prime geodesics and cusps in $G\backslash\mathbb{H}$ mapping to the same prime geodesic or cusp, respectively, in $\Gamma\backslash\mathbb{H}$ through the quotient map. Moreover, elements in $H(G) \cup P(G)$ generate their own centralizer in G, i.e., we have

$$\gamma \in H(G) \cup P(G) \Rightarrow G_\gamma = \langle\gamma\rangle.$$

In this section we introduce the hyperbolic regularization of the trace. The idea is to consider only the hyperbolic elements in the summation occurring in formula (2.2.4).

Definition 2.4.1. Let $G \subseteq \Gamma$ be a subgroup. The *hyperbolic part* of $K_k^G(t; z, w)$ is defined by the formula

$$\mathrm{H}K_k^G(t; z, w) = \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} \sum_{\substack{\gamma \in G \\ \gamma \text{ hyperbolic}}} j(\gamma, z)^k \left(\frac{\gamma(z) - \bar{w}}{w - \gamma(\bar{z})} \right)^k K_k(t; d_{\mathrm{hyp}}(w, \gamma(z))),$$

similarly, its *parabolic part* is defined by

$$\mathrm{P}K_k^G(t; z, w) = \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} \sum_{\substack{\gamma \in G \\ \gamma \text{ parabolic}}} j(\gamma, z)^k \left(\frac{\gamma(z) - \bar{w}}{w - \gamma(\bar{z})} \right)^k K_k(t; d_{\mathrm{hyp}}(w, \gamma(z))).$$

Moreover, the *hyperbolic heat trace* of $K_k^G(t; z, w)$ is

$$\mathrm{HTr}\, K_k^G(t) = \int_{G \backslash \mathbb{H}} \mathrm{H}K_k^G(t; z, z)\, \mu_{\mathrm{hyp}}(z).$$

Proposition 2.4.2. *The on-diagonal hyperbolic part* $\mathrm{H}K_k^G(t; z, z)$ *is a well-defined real valued function on* $\mathbb{R}_{>0} \times G \backslash \mathbb{H}$.

Proof. First we remark that for $k = 0$ and $G = \Gamma$ this proposition is theorem 1.1.a in [39]. Since the inverse of a hyperbolic element is still hyperbolic, the argument proving corollary 2.2.4 ensures that, if convergent, the on-diagonal hyperbolic part is real valued. To prove it is well-defined for $z \in G \backslash \mathbb{H}$ we verify that it is invariant by the simultaneous action of G on the two space variables. Indeed let $\delta \in G$, then

$$\mathrm{H}K_k^G(t; \delta z, \delta z) = \sum_{\substack{\gamma \in G \\ \gamma \text{ hyperbolic}}} j(\gamma, \delta(z))^k \left(\frac{\gamma\delta(z) - \delta(\bar{z})}{\delta(z) - \gamma\delta(\bar{z})} \right)^k K_k(t; d_{\mathrm{hyp}}(\delta(z), \gamma\delta(z))).$$

Using formula (2.2.8), we find

$$\mathrm{H}K_k^G(t; \delta z, \delta z) = \sum_{\substack{\gamma \in G \\ \gamma \text{ hyperbolic}}} \left(\frac{j(\gamma, \delta(z)) j(\delta, z)}{j(\delta, \delta^{-1}\gamma\delta(z))} \right)^k \left(\frac{\delta^{-1}\gamma\delta(z) - \bar{z}}{z - \delta^{-1}\gamma\delta(\bar{z})} \right)^k$$
$$\times K_k(t; d_{\mathrm{hyp}}(z, \delta^{-1}\gamma\delta(z))).$$

And by formula (2.2.2), we conclude

$$\begin{aligned}\mathrm{H}K_k^G(t; \delta z, \delta z) &= \sum_{\substack{\gamma \in G \\ \gamma \text{ hyperbolic}}} j(\delta^{-1}\gamma\delta, z)^k \left(\frac{\delta^{-1}\gamma\delta(z) - \bar{z}}{z - \delta^{-1}\gamma\delta(\bar{z})} \right)^k K_k(t; d_{\mathrm{hyp}}(z, \delta^{-1}\gamma\delta(z))) \\ &= \mathrm{H}K_k^G(t; z, z).\end{aligned}$$

To complete the proof it remains to show that the sum is convergent for any $t \in \mathbb{R}_{>0}$, $z \in G \backslash \mathbb{H}$ and $k \geq 0$. Without loss of generality we assume $G = \Gamma$. For $z \in \mathcal{F}_\Gamma$ and $d \in \mathbb{R}_{>0}$ let

$$\mathrm{N}_\Gamma(z, d) = \left| \{ \gamma \in \Gamma \mid d_{\mathrm{hyp}}(z, \gamma(z)) < d \} \right|.$$

Equivalently, $\mathrm{N}_\Gamma(z,d)$ is the number of geodesic paths on $Y(\Gamma)$ connecting $z \in Y(\Gamma)$ to itself and having length at most d. Let us also recall that the injectivity radius $i(z)$ of $z \in Y(\Gamma)$ is the distance of z from the closest point $w \in Y(\Gamma)$ such that there are two minimizing geodesics connecting z and w, i.e., it is half of the length of the shortest closed geodesic passing through z. The next lemma is a classic estimate on the quantity $\mathrm{N}_\Gamma(z,d)$.

Lemma 2.4.3. *The following estimate holds*

$$\mathrm{N}_\Gamma(z,d) \ll_{i(z)} e^d.$$

Proof. By definition of injectivity radius, the distance of two distinct elements in the Γ-orbit of z on $\mathbb{H}$ has to be at least $i(z)$. Since the area of a hyperbolic disk of radius r is $2\pi(\cosh(r)-1)$, we find

$$\mathrm{N}_\Gamma(z,d) \le \frac{2\pi(\cosh(d+i(z))-1)}{2\pi(\cosh(i(z))-1)} \ll_{i(z)} e^d. \tag{2.4.2}$$

□

Using this lemma we prove the convergence of the on-diagonal hyperbolic part. We first compute

$$\left|\mathrm{H}K_k^\Gamma(t;z,z)\right| \le \sum_{\substack{\gamma\in\Gamma \\ \gamma \text{ hyperbolic}}} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))).$$

Rewriting the sum as a Stieltjes integral, we deduce

$$\left|\mathrm{H}K_k^\Gamma(t;z,z)\right| \le \int_0^\infty K_k(t;v)\,\mathrm{dN}_\Gamma(z,v).$$

We apply lemma 2.4.3 and proposition 2.3.4, then

$$\left|\mathrm{H}K_k^\Gamma(t;z,z)\right| \ll_{i(z),k} \int_0^\infty e^{v-\frac{v^2}{4t}}\,\mathrm{d}v.$$

This proves the proposition. □

Let $\gamma \in \Gamma$ be hyperbolic. Then the hyperbolic heat trace associated to $G = \langle\gamma\rangle$ can be written in terms of the length

$$\ell(\gamma) := \inf_{z\in\mathbb{H}} d_{\mathrm{hyp}}(z,\gamma(z))$$

of the closed geodesic on $\Gamma\backslash\mathbb{H}$ associated to $\gamma \in \Gamma$.

Proposition 2.4.4. *Let γ be hyperbolic, then*

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = \sum_{n=1}^\infty \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{n^2\,\ell(\gamma)^2}{4t}}}{2\sqrt{\pi t}}.$$

Proof. With minor modifications, the proof is the repetition of a computation sketched by D'Hoker–Phong on pages 540-541 of [17], who extended the original result of Selberg [56] to higher weights. Whenever possible we maintain the original notation. According to McKean [45, page 229], every hyperbolic element is conjugated in $\mathrm{PSL}_2(\mathbb{R})$ to a magnification, i.e., there exists $\tau \in \mathrm{PSL}_2(\mathbb{R})$ such that $\delta = \tau^{-1}\gamma\tau = \left(\begin{smallmatrix} m & 0 \\ 0 & m^{-1} \end{smallmatrix}\right)$ for $m > 0$. Moreover, we have

$$\mathrm{HTr}\, K_k^{\langle\delta\rangle}(t) = \mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t).$$

This can be verified expanding the left hand side of the relation above, replacing δ by $\tau^{-1}\gamma\tau$ and observing that a fundamental domain $\mathcal{F}_{\langle\gamma\rangle}$ for $\langle\gamma\rangle\backslash\mathbb{H}$ is of the form $\tau\left(\mathcal{F}_{\langle\delta\rangle}\right)$ where $\mathcal{F}_{\langle\delta\rangle}$ is a fundamental domain for $\langle\delta\rangle\backslash\mathbb{H}$, and that

$$j(\tau^{-1}\gamma^n\tau, z)\,\frac{\tau^{-1}\gamma^n\tau z - \bar{z}}{z - \tau^{-1}\gamma^n\tau\bar{z}} = j(\gamma^n, \tau z)\,\frac{\gamma^n\tau z - \tau\bar{z}}{\tau z - \gamma^n\tau\bar{z}},$$

where the last relation is obtained by formula (2.2.8).
Therefore, without loss of generality, we can assume γ to be of the form $\left(\begin{smallmatrix} m & 0 \\ 0 & m^{-1} \end{smallmatrix}\right)$. Since $\ell(\gamma) = \log(m^2)$ and a fundamental domain for the generated subgroup is

$$\mathcal{F}_{\langle\gamma\rangle} = \left\{z = x + iy \in \mathbb{H} \,|\, x \in \mathbb{R},\, y \in (1, m^2)\right\},$$

its associated hyperbolic heat trace has the expression

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = 2\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\int_{1}^{e^{\ell(\gamma)}}\left(\frac{e^{n\,\ell(\gamma)}z - \bar{z}}{z - e^{n\,\ell(\gamma)}\bar{z}}\right)^k K_k\left(t; d_{\mathrm{hyp}}\left(z, e^{n\,\ell(\gamma)}z\right)\right)\frac{\mathrm{d}y}{y^2}\,\mathrm{d}x.$$

On this expression we perform the change of variables $u = \frac{x}{y}$, $v = xy$. The differentials change according to $\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{\mathrm{d}v\mathrm{d}u}{2v}$, and the domain of integration transforms according to

$$\left\{x \in \mathbb{R},\, y \in \left(1, e^{\ell(\gamma)}\right)\right\} \longmapsto \left\{u \in \mathbb{R},\, v \in \left(u, u\,e^{2\ell(\gamma)}\right)\right\}.$$

Moreover, the hyperbolic distance

$$d(u, n, \gamma) := d_{\mathrm{hyp}}\left(z, e^{n\,\ell(\gamma)}z\right) = \operatorname{arccosh}\left(2u^2\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)^2 + \cosh(n\,\ell(\gamma))\right) \qquad (2.4.3)$$

and the expression

$$\frac{e^{n\,\ell(\gamma)}z - \bar{z}}{z - e^{n\,\ell(\gamma)}\bar{z}} = \frac{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) - iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) + iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}$$

are only dependent on u. Thus, we immediately perform the v-integration, which yields the factor $2\ell(\gamma)$. If, in addition, we explicit the form of the heat kernel $K_k\left(t; d_{\mathrm{hyp}}\left(z, e^{n\,\ell(\gamma)}z\right)\right)$, we find

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = \sum_{n=1}^{\infty} \frac{\ell(\gamma)\,\sqrt{2}e^{-t\left(k+\frac{1}{2}\right)^2}}{4(\pi t)^{\frac{3}{2}}} \int\limits_{-\infty}^{\infty} \left(\frac{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) - iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) + iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \right)^k$$
$$\times \int\limits_{d(u,n,\gamma)}^{\infty} \frac{be^{-\frac{b^2}{4t}}}{\sqrt{\cosh(b) - \cosh(d(u,n,\gamma))}} T_{2k}\left(\frac{\cosh\left(\frac{b}{2}\right)}{\cosh\left(\frac{d(u,n,\gamma)}{2}\right)} \right) \mathrm{d}b\,\mathrm{d}u.$$

We exchange the order of the b- and the u-integration. To justify this, observe that, by definition of $\ell(\gamma)$, we have

$$d(u,n,\gamma) \geq \ell(\gamma) \qquad (u \in \mathbb{R},\, n \in \mathbb{N}_{\geq 1})\,.$$

Therefore, we can apply proposition 2.3.4 with $\delta = \ell(\gamma)$, to obtain

$$\int\limits_{d(u,n,\gamma)}^{\infty} \frac{be^{-\frac{b^2}{4t}}}{\sqrt{\cosh(b) - \cosh(d(u,n,\gamma))}} T_{2k}\left(\frac{\cosh\left(\frac{b}{2}\right)}{\cosh\left(\frac{d(u,n,\gamma)}{2}\right)} \right) \mathrm{d}b \ll_{\gamma,k,t} e^{-\frac{d(u,n,\gamma)^2}{4t}}.$$

Thus, the absolute value of the integrand of the u-integral admits the same bound. Using formula (2.4.3), and the elementary bound $\operatorname{arccosh}(Z+1) \geq \log(Z+1)$ for $Z \geq 0$, we estimate

$$\begin{aligned} d(u,n,\gamma) &\geq \operatorname{arccosh}\left(2u^2 \sinh\left(\frac{n\,\ell(\gamma)}{2}\right)^2 + 1 \right) \\ &\geq \log\left(2u^2 \sinh\left(\frac{n\,\ell(\gamma)}{2}\right)^2 + 1 \right) \sim 2\log(|u|) + O_{n,\gamma}(1) \qquad (u \to \pm\infty)\,. \end{aligned}$$

We can now apply Fubini's theorem and exchange the integrals. Changing the domains of integration accordingly, we find

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = \sum_{n=1}^{\infty} \frac{\ell(\gamma)\,\sqrt{2}e^{-t\left(k+\frac{1}{2}\right)^2}}{4(\pi t)^{\frac{3}{2}}} \int\limits_{n\,\ell(\gamma)}^{\infty} be^{-\frac{b^2}{4t}}$$
$$\times \int\limits_{-\frac{\sqrt{\cosh(b)-\cosh(n\,\ell(\gamma))}}{\sqrt{2}\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}}^{\frac{\sqrt{\cosh(b)-\cosh(n\,\ell(\gamma))}}{\sqrt{2}\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}} \left(\frac{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) - iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) + iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \right)^k \frac{T_{2k}\left(\frac{\cosh\left(\frac{b}{2}\right)}{\cosh\left(\frac{d(u,n,\gamma)}{2}\right)} \right)}{\sqrt{\cosh(b) - \cosh(d(u,n,\gamma))}} \mathrm{d}u\,\mathrm{d}b.$$

The u-integral can be evaluated separately, at the same time we show it is independent of b. To do so we change again variables to

$$\theta = \arcsin\left(\frac{\sqrt{2}u\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\sqrt{\cosh(b) - \cosh(n\,\ell(\gamma))}} \right).$$

And, to ease the notation, we write

$$B = \frac{\sqrt{2}\cosh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\sqrt{\cosh(b) - \cosh(n\,\ell(\gamma))}}.$$

Multiplying and dividing by $\frac{\sqrt{\cosh(b)-\cosh(n\,\ell(\gamma))}}{\sqrt{2}}$, we find

$$\frac{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) - iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}{\cosh\left(\frac{n\,\ell(\gamma)}{2}\right) + iu\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} = \frac{B - i\sin(\theta)}{B + i\sin(\theta)}.$$

Also

$$\frac{\cosh\left(\frac{b}{2}\right)}{\cosh\left(\frac{d(u,n,\gamma)}{2}\right)} = \sqrt{\frac{\cosh(b) + 1 + \cosh(n\,\ell(\gamma)) - \cosh(n\,\ell(\gamma))}{2u^2\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)^2 + \cosh(n\,\ell(\gamma)) + 1}} = \frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}},$$

and

$$\frac{\sqrt{\cosh(b) - \cosh(n\,\ell(\gamma))}\,\cos(\theta)}{\sqrt{\cosh(b) - \cosh(d(u,n,\gamma))}} = \sqrt{\frac{(\cosh(b) - \cosh(n\,\ell(\gamma)))\,\cos(\theta)^2}{(\cosh(b) - \cosh(n\,\ell(\gamma)))\,(1 - \sin(\theta)^2)}} = 1.$$

Using these relations, the u-integral transforms into

$$\frac{1}{\sqrt{2}\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{B - i\sin(\theta)}{B + i\sin(\theta)}\right)^k T_{2k}\left(\frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}}\right) \mathrm{d}\theta.$$

Equation 18.12.8 of [48] gives a generating function for Chebyshev polynomials

$$\frac{1 - XZ}{1 - 2XZ + Z^2} = \sum_{n=0}^{\infty} T_n(X) Z^n \qquad (X, Z \in \mathbb{R}).$$

Using it, we deduce

$$\sum_{m=0}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda^m \left(\sqrt{\frac{B - i\sin(\theta)}{B + i\sin(\theta)}}\right)^m T_m\left(\frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}}\right) \mathrm{d}\theta$$
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(B - i\sin(\theta) - \lambda\sqrt{B^2+1})}{(B - i\sin(\theta)) - 2\lambda\sqrt{B^2+1} + \lambda^2(B + i\sin(\theta)} \mathrm{d}\theta$$

where $\lambda \in \left(0, \min\left\{1, \frac{(B+1)\sqrt{B-1}}{B^{\frac{3}{2}}}\right\}\right)$ is a parameter ensuring absolute convergence of the series. The real part of the latter integral is π-periodic, while its imaginary part is odd and 2π-periodic; thus we extend the domain of integration to $(-\pi, \pi)$ multiplying by $\frac{1}{2}$.

With another change of variables, $\zeta = e^{i\theta}$, we express the integral as a contour integral around the unit circle, and find

$$\sum_{m=0}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda^m \left(\sqrt{\frac{B - i\sin(\theta)}{B + i\sin(\theta)}}\right)^m T_m \left(\frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}}\right) \mathrm{d}\theta$$
$$= \frac{1}{2} \int_{\partial B_1(0)} \frac{2B\zeta - \zeta^2 + 1 - 2\lambda\sqrt{B^2+1}\zeta}{2B\zeta - \zeta^2 + 1 - 4\lambda\sqrt{B^2+1}\zeta + 2\lambda^2 B\zeta + \lambda^2\zeta^2 - \lambda^2} \frac{\mathrm{d}\zeta}{i\zeta}.$$

The two poles of the integrand can be explicitly computed. They are

$$\zeta_1 = 0, \qquad \zeta_2 = \frac{(1+\lambda)^2}{1-\lambda^2}\left(B - \sqrt{B^2+1}\right),$$

and they both lie in the unit circle for λ in the chosen domain. Finally we apply the residue theorem: From ζ_1 we obtain the contribution $\frac{\pi}{1-\lambda^2}$ and from ζ_2 the contribution $\frac{\pi\lambda}{1-\lambda^2}$. Thus, we obtain

$$\sum_{m=0}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \lambda^m \left(\sqrt{\frac{B - i\sin(\theta)}{B + i\sin(\theta)}}\right)^m T_m \left(\frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}}\right) \mathrm{d}\theta = \frac{\pi}{1-\lambda}.$$

From the summation formula of the geometric series, this implies

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{B - i\sin(\theta)}{B + i\sin(\theta)}\right)^k T_{2k} \left(\frac{\sqrt{B^2+1}}{\sqrt{B^2 + \sin(\theta)^2}}\right) \mathrm{d}\theta = \pi.$$

We have now completed the evaluation of the u-integral, and proved that it is independent of b. Replacing its value into the expression of the hyperbolic heat trace, we find

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = \sum_{n=1}^{\infty} \frac{\pi\,\ell(\gamma)\, e^{-t\left(k+\frac{1}{2}\right)^2}}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right) 4(\pi t)^{\frac{3}{2}}} \int_{n\,\ell(\gamma)}^{\infty} b e^{-\frac{b^2}{4t}} \mathrm{d}b.$$

Explicitly performing the remaining integration gives the desired result. □

Proposition 2.4.5. *Let $G \subseteq \Gamma$ be a subgroup, then*

$$\mathrm{HTr}\, K_k^{G}(t) = \frac{1}{2} \sum_{\gamma \in H(G)} \mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t).$$

Proof. For $k = 0$ and $G = \Gamma$ this proposition is theorem 1.1.b in [39]. By definition of hyperbolic heat trace and formula (2.4.1), we have

$$\mathrm{HTr}\, K_k^G(t) = \sum_{\gamma \in H(G)} \int_{G\backslash\mathbb{H}} \sum_{[\eta]\in G_\gamma\backslash G} \sum_{n=1}^{\infty} j(\eta^{-1}\gamma^n\eta, z)^k \left(\frac{\eta^{-1}\gamma^n\eta(z)-\bar{z}}{z-\eta^{-1}\gamma^n\eta(\bar{z})}\right)^k$$
$$\times K_k(t; d_{\mathrm{hyp}}(z, \eta^{-1}\gamma^n\eta(z)))\,\mu_{\mathrm{hyp}}(z).$$

Using formulae (2.2.2) and (2.2.8) as in the proof of proposition 2.4.2, we find

$$\mathrm{HTr}\, K_k^G(t) = \sum_{\gamma \in H(G)} \int_{G\backslash\mathbb{H}} \sum_{[\eta]\in G_\gamma\backslash G} \sum_{n=1}^{\infty} j(\gamma^n, \eta(z))^k \left(\frac{\gamma^n\eta(z)-\eta(\bar{z})}{\eta(z)-\gamma^n\eta(\bar{z})}\right)^k$$
$$\times K_k(t; d_{\mathrm{hyp}}(\eta(z), \gamma^n\eta(z)))\,\mu_{\mathrm{hyp}}(z).$$

Observe that, by primitivity, $G_\gamma = \langle\gamma\rangle$. Formally unfolding the integration domain along the cosets of G_γ in G, i.e., using the isometries to the respective fundamental domains and unfolding the latter ones, gives

$$\mathrm{HTr}\, K_k^G(t) = \sum_{\gamma \in H(G)} \int_{\langle\gamma\rangle\backslash\mathbb{H}} \sum_{n=1}^{\infty} j(\gamma^n, z)^k \left(\frac{\gamma^n z-\bar{z}}{z-\gamma^n\bar{z}}\right)^k K_k(t; d_{\mathrm{hyp}}(z, \gamma^n(z)))\,\mu_{\mathrm{hyp}}(z).$$

We chose elements in $H(G)$ in such a way that if $\gamma \in H(G)$ also γ^{-1} does. Since $\langle\gamma\rangle = \langle\gamma^{-1}\rangle \simeq \mathbb{Z}$, we multiply by $\frac{1}{2}$ and extend the summation to the negative integers. The result is

$$\mathrm{HTr}\, K_k^G(t) = \frac{1}{2} \sum_{\gamma \in H(G)} \int_{\langle\gamma\rangle\backslash\mathbb{H}} \left(K_k^{\langle\gamma\rangle}(t; z, z) - K_k(t; 0)\right) \mu_{\mathrm{hyp}}(z).$$

The claim now follows because the only non-hyperbolic element in $\langle\gamma\rangle$ is the identity. □

To close the section, we prove that hyperbolic heat traces are always convergent.

Proposition 2.4.6. *For $G \subseteq \Gamma$ a subgroup and $t \in \mathbb{R}_{>0}$ fixed, it holds*

$$\mathrm{HTr}\, K_k^G(t) < \infty.$$

Proof. We preliminary observe that, with arguments similar to the upcoming lemma 2.5.2, it is possible to provide a space-uniform upper bound for $\mathrm{H}K_k^G(t; z, z)$. Then the proposition would follow immediately for all the cofinite subgroups $G \subseteq \Gamma$. To avoid the cofiniteness hypothesis we argument differently. First we provide an estimate, depending on $\ell(\gamma)$, for $\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t)$. By proposition 2.4.4 we have

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) = \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{n^2\,\ell(\gamma)^2}{4t}}}{2\sqrt{\pi t}}.$$

Since $(v+1)\sinh(v) \gg ve^v$ for $v > 0$, we have the inequality

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) \ll \frac{e^{-tk(k+1)}}{2\sqrt{\pi t}} \sum_{n=1}^{\infty} \left(\ell(\gamma) + \frac{1}{n} \right) e^{-\frac{(n\,\ell(\gamma)+t)^2}{4t}}.$$

Bounding the sum for $n \geq 2$ by the corresponding integral and changing variables, we deduce

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) \ll (\ell(\gamma)+1) \frac{e^{-tk(k+1)}}{2\sqrt{\pi t}} \left(e^{-\frac{(\ell(\gamma)+t)^2}{4t}} + \frac{2\sqrt{t}}{\ell(\gamma)} \int_{\frac{\ell(\gamma)+t}{2\sqrt{t}}}^{\infty} e^{-u^2} \mathrm{d}u \right).$$

The latter integral is a complementary error function. Formula (5) of [15] gives the bound

$$\mathrm{erfc}(v) \leq e^{-v^2} \qquad (v > 0)\,.$$

Applying it, we find

$$\mathrm{HTr}\, K_k^{\langle\gamma\rangle}(t) \ll \frac{\ell(\gamma)+1}{2\sqrt{\pi t}} \left(\frac{2\sqrt{t}}{\ell(\gamma)} + 1 \right) e^{-t\left(k+\frac{1}{2}\right)^2} e^{-\frac{\ell(\gamma)^2}{4t}}.$$

To complete the proposition, we quote

$$\pi_{00,\Gamma}(v) = |\{\gamma \in \Gamma \,|\, \gamma \text{ hyperbolic }, \ell(\gamma) < v\}| = O_\Gamma(e^v)$$

from page 475 of [33]. This immediately implies

$$\pi_{00,G}(v) = |\{\gamma \in G \,|\, \gamma \text{ hyperbolic }, \ell(\gamma) < v\}| = O_G(e^v).$$

Let $d_G \in \mathbb{R}_{>0}$ such that there exists at least one hyperbolic element $\gamma \in G$ such that $\ell(\gamma) < d_G$. By the last formula, there are only finitely many hyperbolic elements $\gamma \in G$ such that $\ell(\gamma) < d_G$. Since for each such γ we have $\ell(\gamma) > 0$, we find

$$\begin{aligned} \inf\{\ell(\gamma) \,|\, \gamma \in G,\, \gamma \text{ hyperbolic }\} &= \inf\{\ell(\gamma) \,|\, \gamma \in G,\, \gamma \text{ hyperbolic }, \ell(\gamma) < d_G\} \\ &= \min\{\ell(\gamma) \,|\, \gamma \in G,\, \gamma \text{ hyperbolic }, \ell(\gamma) < d_G\} > 0. \end{aligned}$$

Now, since $H(G)$ is a subset of the set of hyperbolic elements of G, we have

$$\inf\{\ell(\gamma) \,|\, \gamma \in H(G)\} \geq \inf\{\ell(\gamma) \,|\, \gamma \in G,\, \gamma \text{ hyperbolic }\} > 0. \qquad (2.4.4)$$

By the last two expressions and proposition 2.4.5 the result is proven. □

2.5 The cusp regularization of the trace of the heat kernel

In this section we prove that the hyperbolic heat trace equals a differently defined and more geometrically meaningful regularization of the trace, which is obtained by ignoring the contribution to the heat kernel of the stabilizer of a cusp in proximity of the cusp itself. Also this section is a generalization of ideas from [39] to higher weights.

Let P_j be a cusp of X and $\gamma_j \in P(\Gamma)$ a generator of its stabilizer. There exists $\sigma_j \in \mathrm{PSL}_2(\mathbb{R})$ such that $\sigma_j(i\infty) = P_j$ and $\sigma_j^{-1}\gamma_j\sigma_j = \left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$; the matrix σ_j is called *scaling matrix of the cusp* P_j. In general, it will be easier to deal with $i\infty$, its stabilizer $\Gamma_\infty := \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$, generated by $\gamma_\infty := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and its associated heat kernel

$$K_k^\infty(t; z, w) := K_k^{\Gamma_\infty}(t; z, w) = \frac{\mathrm{Im}(z)^k}{\mathrm{Im}(w)^k} \sum_{n\in\mathbb{Z}} \left(\frac{z+n-\bar{w}}{w-\bar{z}-n}\right)^k K_k(t; d_{\mathrm{hyp}}(z+n, w)),$$

rather than with P_j, its stabilizer $\langle\gamma_j\rangle$ and its associated heat kernel $K_k^{\langle\gamma_j\rangle}(t; z, w)$. A fundamental domain for the quotient $\Gamma_\infty\backslash\mathbb{H}$ is the strip

$$\mathcal{F}_\infty := \{z = x + iy \in \mathbb{H} \,|\, 0 \leq x < 1,\, y > 0\}.$$

Moreover, for $0 \leq a < b \leq \infty$, we consider the truncated fundamental domain

$$\mathcal{F}_\infty(a < y < b) := \{z = x + iy \in \mathcal{F}_\infty \,|\, a < y < b\},$$

where we omit a and b from the writing if they are equal to 0 or ∞, respectively. Observing that the exponential map $z \mapsto q = e^{2\pi i z}$ maps the cusp $i\infty$ to the origin $q = 0$, we find that the punctured disc $\{q \in \mathbb{C}^\times \,|\, |q| < \epsilon\}$ of radius $\epsilon \in \mathbb{R}_{>0}$ corresponds to the punctured neighborhood

$$B_\epsilon(i\infty) = \mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi} < y\right)$$

of the cusp $i\infty$. In general, we obtain the punctured neighborhood

$$B_\epsilon(P_j) := \sigma_j\left(\mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi} < y\right)\right)$$

of the cusp P_j. The hyperbolic volume of $B_\epsilon(P_j)$ is given by $-\frac{2\pi}{\log(\epsilon)}$. Before stating the claimed characterization of the hyperbolic heat trace, which generalizes theorem 1.1.c of [39] to $k \geq 1$, we need to further examine the quantity $\mathrm{N}_\Gamma(z, d)$. Specifically, the estimate given by formula (2.4.2) blows up for $y \to \infty$. If y is large enough, the points w close to z for which there are two minimizing geodesics connecting z and w are of the form $w = z + \frac{1}{2}$, since the cusp $i\infty$ has width 1. By definition, the injectivity radius giving the distance from z to the closest of these points equals

$$i(z) = \min_{v\in\mathbb{R}_{>0}} d_{\mathrm{hyp}}\left(iy, \frac{1}{2} + iv\right).$$

Since

$$d_{\mathrm{hyp}}\left(iy, \frac{1}{2} + iv\right) = \operatorname{arccosh}\left(1 + \frac{\left|iy - \left(\frac{1}{2} + iv\right)\right|^2}{2yv}\right) = \operatorname{arccosh}\left(1 + \frac{(y-v)^2 + \frac{1}{4}}{2yv}\right),$$

the minimum is checked to be attained for $v = \sqrt{y^2 + \frac{1}{4}}$. Thus, one gets

$$
\begin{aligned}
i(z) &= \operatorname{arccosh}\left(1+\frac{\left(y-\sqrt{y^2+\frac14}\right)^2+\frac14}{2y\sqrt{y^2+\frac14}}\right) = \operatorname{arccosh}\left(1+\frac{2\left(y^2+\frac14\right)-2y\sqrt{y^2+\frac14}}{2y\sqrt{y^2+\frac14}}\right) \\
&= \operatorname{arccosh}\left(\frac{2\left(y^2+\frac14\right)}{2y\sqrt{y^2+\frac14}}\right) = \operatorname{arccosh}\left(\frac{\sqrt{y^2+\frac14}}{y}\right) \underset{y\to\infty}{\longrightarrow} \operatorname{arccosh}(1) = 0. \qquad (2.5.1)
\end{aligned}
$$

To see the blow up we substitute the explicit value into the expression for $\mathrm{N}_\Gamma(z,d)$ given by formula (2.4.2). We find

$$
\mathrm{N}_\Gamma(z,d) \leq \frac{2\pi(\cosh(d+i(z))-1)}{2\pi(\cosh(i(z))-1)} = \frac{\cosh(d)\cosh(iz)+\sinh(d)\sinh(i(z))-1}{\cosh(i(z))-1}.
$$

Now, using the explicit expression for the injectivity radius of formula (2.5.1), we have

$$
\cosh(i(z)) = \frac{\sqrt{y^2+\frac14}}{y} = \sqrt{1+\frac{1}{4y^2}} \qquad (y \gg_\Gamma 1)\,.
$$

Since we are looking at the behavior for y large we can harmlessly assume $y>1$, then we obtain

$$
\cosh(i(z)) = \sqrt{1+\frac{1}{4y^2}} \geq 1 \geq \frac12\left(1+\frac{1}{y^2}\right) \gg 1+\frac{1}{y^2}.
$$

This gives the inequality

$$
\frac{1}{\cosh(i(z))-1} \ll y^2 \qquad (y \gg_\Gamma 1)\,,
$$

which in turn implies the estimate

$$
\mathrm{N}_\Gamma(z,d) \ll y^2\, e^d \qquad (y \gg_\Gamma 1)\,. \tag{2.5.2}
$$

To improve this estimate, we want to take into account the fact, that we are not considering the elements belonging to the stabilizer of the cusp under consideration. We aim to prove a bound for

$$
\mathrm{N}_{\Gamma\setminus\Gamma_\infty}(z,d) := |\{\gamma\in\Gamma\setminus\Gamma_\infty \,|\, d_{\mathrm{hyp}}(z,\gamma(z))<d\}|\,.
$$

To do so, we first quote, without proof, lemma 1.25 of [57].

Lemma 2.5.1. *There exists a positive number r_Γ, depending only on Γ, such that*

$$
\operatorname{Im}(z)\operatorname{Im}(\gamma(z)) \leq \frac{1}{r_\Gamma^2} \qquad (z\in\mathbb{H},\ \gamma\in\Gamma\setminus\Gamma_\infty)\,. \qquad \square
$$

Using the quantity r_Γ defined by the last lemma, we estimate the quantity $\mathrm{N}_{\Gamma\setminus\Gamma_\infty}(z,d)$.

Lemma 2.5.2. *Let $z \in \mathbb{H}$ be such that $y \geq \frac{2}{r_\Gamma}$ and $y > 1$. We write $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right)$. Then, we have*

$$\mathrm{N}_{\Gamma\backslash\Gamma_\infty}(z,d) \ll_\Gamma \begin{cases} 0, & d < a_\Gamma(z), \\ y\, e^d, & d \geq a_\Gamma(z). \end{cases}$$

Proof. The idea of the proof is to observe that, if y is large, the images of z under the action of elements $\gamma, \delta \in \Gamma \setminus \Gamma_\infty$ lying in distinct cosets in Γ/Γ_∞ have to be far from each other and far from z. We can therefore split the problem into two parts: On the one hand, we count cosets in Γ/Γ_∞ having at least an element with distance smaller than d to z, on the other hand we use the distance of the closest element to z in a coset in Γ/Γ_∞ to bound the number of elements in the same coset with distance at most d from z.
The first part of the proof consists in proving a bound for

$$\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d) := \left|\{[\gamma] \in \Gamma/\Gamma_\infty, [\gamma] \neq \Gamma_\infty \,|\, \exists\, \widetilde{\gamma} \in [\gamma] \,:\, d_{\mathrm{hyp}}(z, \widetilde{\gamma}(z)) < d\}\right|.$$

We aim at proving a lower bound for $d_{\mathrm{hyp}}(z,\gamma(z))$ if $\gamma \in \Gamma \setminus \Gamma_\infty$. The formula (2.1.1) of the hyperbolic distance function implies

$$d_{\mathrm{hyp}}(z,w) \geq d_{\mathrm{hyp}}(i\mathrm{Im}(z), i\mathrm{Im}(w)) \qquad (w \in \mathbb{H}).$$

Moreover, by lemma 2.5.1, and by the assumption $y \geq \frac{2}{r_\Gamma}$, we have the inequalities

$$\mathrm{Im}(\gamma(z)) \leq \frac{1}{r_\Gamma^2 \mathrm{Im}(z)} \leq \frac{1}{2r_\Gamma} \leq \mathrm{Im}(z) \qquad (\gamma \in \Gamma \setminus \Gamma_\infty).$$

Therefore, we find

$$d_{\mathrm{hyp}}(z,\gamma(z)) \geq d_{\mathrm{hyp}}(i\mathrm{Im}(z), i\mathrm{Im}(\gamma(z))) \geq d_{\mathrm{hyp}}\left(iy, \frac{i}{r_\Gamma^2 y}\right) \qquad (\gamma \in \Gamma \setminus \Gamma_\infty). \tag{2.5.3}$$

Using the last relations, the assumption $y \geq \frac{2}{r_\Gamma}$ and the inequality $\operatorname{arccosh}(Z) \geq \log(Z)$ valid for $Z \geq 1$, we provide a lower bound for $d_{\mathrm{hyp}}(z,\gamma(z))$ if $\gamma \in \Gamma \setminus \Gamma_\infty$. We compute

$$\begin{aligned} d_{\mathrm{hyp}}(z,\gamma(z)) &\geq d_{\mathrm{hyp}}\left(iy, \frac{i}{r_\Gamma^2 y}\right) = \operatorname{arccosh}\left(1 + \frac{\left|y - \frac{1}{r_\Gamma^2 y}\right|^2}{2y\frac{1}{r_\Gamma^2 y}}\right) = \operatorname{arccosh}\left(\frac{r_\Gamma^2 y^2 + \frac{1}{r_\Gamma^2 y^2}}{2}\right) \\ &\geq \operatorname{arccosh}\left(\frac{r_\Gamma^2 y^2}{2}\right) \geq \log\left(\frac{r_\Gamma^2 y^2}{2}\right) = a_\Gamma(z) \qquad (\gamma \in \Gamma \setminus \Gamma_\infty). \end{aligned}$$

A first consequence of the last computation is that for $d < a_\Gamma(z)$, we have

$$\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d) = 0. \tag{2.5.4}$$

A second consequence of the last computation is the following: Let $\gamma, \delta \in \Gamma \setminus \Gamma_\infty$ be such that they generate different left cosets in Γ/Γ_∞, i.e., $\gamma^{-1}\delta \notin \Gamma_\infty$. Then, we have

$$d_{\mathrm{hyp}}(\gamma(z), \delta(z)) = d_{\mathrm{hyp}}(z, \gamma^{-1}\delta(z)) \geq a_\Gamma(z).$$

Therefore, since the volume of a hyperbolic ball of radius r is $2\pi(\cosh(r)-1)$, we can bound $\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d)$ by the ratio of the volume of the hyperbolic ball of radius $d+a_\Gamma(z)$ and the volume of the hyperbolic ball of radius $a_\Gamma(z)$, as in the proof of lemma 2.4.3. In formulae, we have

$$\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d) \le \frac{\cosh\left(d+a_\Gamma(z)\right)-1}{\cosh\left(a_\Gamma(z)\right)-1}.$$

From the latter inequality we want to obtain an estimate for $\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d)$ whose dependence on z is not implicit. To do so, let us observe that, for $W\ge c>0$ uniformly bounded from below, we have the estimate

$$\frac{\cosh(Z+W)}{\cosh(W)} = \frac{e^Z+e^{-Z-2W}}{1+e^{-2W}} \ll_c e^Z \qquad (Z\ge 1)\,.$$

We thus need a positive uniform lower bound for $a_\Gamma(z)$. Since $y\ge \frac{2}{r_\Gamma}$, we compute

$$a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right) \ge \log(2).$$

Combining the last three formulae we deduce the estimate on $\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d)$ that completes the first part of the proof. It reads

$$\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,d) \ll e^d. \tag{2.5.5}$$

We remark that equation (2.5.5) provides a bound on the number of cosets in Γ/Γ_∞ that could contain elements with distance smaller than d to z. The second part of the proof consists in estimating how many elements in a fixed coset $[\gamma]\in\Gamma/\Gamma_\infty$ have distance at most d from z. Observe that, for γ fixed, there is a natural bijection between $[\gamma]$ and $\mathbb{Z}$, because elements $\delta\in[\gamma]$ are of the form $\delta=\gamma\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)\in[\gamma]$ for some $n\in\mathbb{Z}$.
Let $\delta_0\in[\gamma]$ be such that it realizes the minimum of $d_{\mathrm{hyp}}(z,\gamma(z))$ in its coset $[\gamma]\in\Gamma/\Gamma_\infty$, i.e., it fulfills the inequality

$$d_{\mathrm{hyp}}(z,\delta(z)) \ge d_{\mathrm{hyp}}(z,\delta_0(z)) \qquad (\delta\in[\gamma])\,.$$

To further characterize δ_0, we compute $d_{\mathrm{hyp}}(z,\delta(z))$ for $\delta=\gamma\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right)$. Applying formula (2.1.1) we obtain

$$\begin{aligned} d_{\mathrm{hyp}}(z,\delta(z)) &= d_{\mathrm{hyp}}(z,\gamma(z+n)) = d_{\mathrm{hyp}}(\gamma^{-1}(z)-n,z)\\ &= \operatorname{arccosh}\left(\frac{(\mathrm{Re}(\gamma^{-1}(z))-\mathrm{Re}(z)-n)^2+y^2+\mathrm{Im}(\gamma^{-1}(z))^2}{2y\mathrm{Im}(\gamma^{-1}(z))}\right). \end{aligned}$$

Observe that the only term of the latter expression depending on n is the contribution of the real parts, which, by an appropriate choice of $n_0\in\mathbb{Z}$, can be bounded by

$$\left|\mathrm{Re}(\gamma^{-1}(z))-\mathrm{Re}(z)-n_0\right|^2 \le \frac{1}{2}.$$

Clearly, the element of $[\gamma]$ corresponding to n_0 in the bijection with $\mathbb{Z}$ is $\delta_0=\gamma\left(\begin{smallmatrix}1&n_0\\0&1\end{smallmatrix}\right)$. To ease notation, and without loss of generality, we assume $n_0=0$, i.e., $\delta_0=\gamma$.

We now provide a lower bound, in terms of n, for the difference of the hyperbolic distances induced by $\delta_n := \delta_0 \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ and δ_0. Proceeding as before, we compute

$$\begin{aligned} d_{\mathrm{hyp}}(z, \delta_n(z)) - d_{\mathrm{hyp}}(z, \delta_0(z)) = &\operatorname{arccosh}\left(\frac{(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z) - n)^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2}{2y\mathrm{Im}(\delta_0^{-1}(z))}\right) \\ &- \operatorname{arccosh}\left(\frac{(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z))^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2}{2y\mathrm{Im}(\delta_0^{-1}(z))}\right). \end{aligned}$$

For $Z \geq 1$ we have the elementary inequalities $\log(Z) \leq \operatorname{arccosh}(Z) \leq \log(2Z)$. Applying them to the first and second term of the last difference, respectively, and writing the difference of the logarithms as the logarithm of the quotient we obtain

$$d_{\mathrm{hyp}}(z, \delta_n(z)) - d_{\mathrm{hyp}}(z, \delta_0(z)) \geq \log\left(\frac{(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z) - n)^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2}{2\left(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z))^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2\right)}\right).$$

We already observed that the minimality of δ_0 implies $\left|\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z)\right| \leq \frac{1}{2}$. Moreover, by lemma (2.5.1) and the assumption $y \geq \frac{2}{r_\Gamma}$, we have

$$\mathrm{Im}(\delta_0^{-1}(z)) \leq \frac{1}{r_\Gamma^2 y} \leq \frac{1}{2r_\Gamma} \leq \frac{y}{4}.$$

These relations imply the inequalities

$$(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z) - n)^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2 \geq (\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z) - n)^2 \geq (|n| - 1)^2,$$

and, since we assumed $y > 1$, one gets

$$2\left(\mathrm{Re}(\delta_0^{-1}(z)) - \mathrm{Re}(z))^2 + y^2 + \mathrm{Im}(\delta_0^{-1}(z))^2\right) \leq 2\left(\frac{1}{2} + y^2 + \frac{y^2}{16}\right) \leq 4y^2.$$

Thus, we proved the inequality

$$d_{\mathrm{hyp}}(z, \delta_n(z)) - d_{\mathrm{hyp}}(z, \delta_0(z)) \geq \log\left(\frac{(|n| - 1)^2}{4y^2}\right),$$

which is the key ingredient to provide an upper bound on the number of $\delta \in [\gamma]$ having a hyperbolic distance smaller than d. Also for the next computation we maintain the notation $\delta_n = \delta_0 \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. We have

$$|\{n \in \mathbb{Z} \,|\, d_{\mathrm{hyp}}(\delta_n, z) < d\}| = |\{n \in \mathbb{Z} \,|\, d_{\mathrm{hyp}}(\delta_n, z) - d_{\mathrm{hyp}}(z, \delta_0(z)) < d - d_{\mathrm{hyp}}(z, \delta_0(z))\}|,$$

therefore

$$\begin{aligned} |\{n \in \mathbb{Z} \,|\, d_{\mathrm{hyp}}(\delta_n, z) < d\}| &\leq \left|\left\{ n \in \mathbb{Z} \,\middle|\, \log\left(\frac{(|n| - 1)^2}{4y^2}\right) < d - d_{\mathrm{hyp}}(z, \delta_0(z)) \right\}\right| \\ &\leq \left|\left\{ n \in \mathbb{Z} \,\middle|\, |n| < 2ye^{\frac{d - d_{\mathrm{hyp}}(z, \delta_0(z))}{2}} + 1 \right\}\right| \ll y\, e^{\frac{d - d_{\mathrm{hyp}}(z, \delta_0(z))}{2}}. \end{aligned}$$

The last step of the proof is to combine the estimates we obtained through a Stieltjes integral. The last estimate implies

$$\mathrm{N}_{\Gamma\backslash\Gamma_\infty}(z,d) \ll \int_0^d y e^{\frac{d-v}{2}} \, \mathrm{d}\mathrm{N}^{\mathrm{cosets}}_{\Gamma\backslash\Gamma_\infty}(z,v).$$

Further applying the estimates (2.5.4) and (2.5.5) we obtain

$$\mathrm{N}_{\Gamma\backslash\Gamma_\infty}(z,d) \ll \int_{a_\Gamma(z)}^d y e^{\frac{d-v}{2}} e^v \, \mathrm{d}v.$$

Explicitly integrating the exponential, and observing that the whole integral is 0 for $d < a_\Gamma(z)$ proves the result. □

Remark 2.5.3. For completeness, we additionally observe, in the obvious notation, the elementary bound

$$\mathrm{N}_{\Gamma_\infty}(z,d) \ll_\Gamma y \, e^{\frac{d}{2}}.$$

Combined with lemma 2.5.2 it yields the following improved version of estimate (2.5.2)

$$\mathrm{N}_\Gamma(z,d) \ll_\Gamma \begin{cases} y \, e^{\frac{d}{2}}, & d < a_\Gamma(z), \\ y \, e^{d}, & d \geq a_\Gamma(z), \end{cases} \qquad (y \gg_\Gamma 1)\,.$$

We can now state the proposition.

Proposition 2.5.4. *For any $\epsilon \in \mathbb{R}_{>0}$ fixed and small enough, we have*

$$\mathrm{HTr}\, K_k^\Gamma(t) = \int_{\mathcal{F}_\Gamma \backslash \bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_k^\Gamma(t;z,z) - K_k(t;0)\right) \mu_{\mathrm{hyp}}(z) \tag{2.5.6}$$

$$+\, p \int_{\mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi} < y\right)} \left(K_k^\Gamma(t;z,z) - K_k^\infty(t;z,z)\right) \mu_{\mathrm{hyp}}(z) \tag{2.5.7}$$

$$-\, p \int_{\mathcal{F}_\infty\left(y < -\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right) \mu_{\mathrm{hyp}}(z). \tag{2.5.8}$$

Proof. To prove the proposition we have to show that each of the three integrals is convergent, and the equality of the formula. We proceed in this order. To begin with, we fix $s_\Gamma \in \mathbb{R}_{>0}$ such that $B_{s_\Gamma}(P_j) \cap B_{s_\Gamma}(P_h) = \emptyset$ for $j \neq h$ and $-\log(s_\Gamma) > 2\pi$, and we restrict to $\epsilon < s_\Gamma$. The integral in (2.5.6) has continuous integrand and compact domain, thus it converges. To prove the convergence of the integral in (2.5.7) we shrink ϵ, if needed, to ensure

$$\mathcal{F}_\Gamma \setminus \mathcal{F}_\infty\left(y < -\frac{\log(\epsilon)}{2\pi}\right) = \mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi} < y\right).$$

Now, given that the integrand is a continuous function and that the domain has finite volume $-\frac{2\pi}{\log(\epsilon)}$, it is enough to show that the integrand is bounded in the limit for $y\to\infty$. We aim to prove a stronger result, namely

$$0 \leq \lim_{y\to\infty} \left| K_k^{\Gamma}(t;z,z) - K_k^{\infty}(t;z,z) \right| \leq \lim_{y\to\infty} \sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))) = 0. \qquad (2.5.9)$$

To prove (2.5.9), we shrink ϵ, if needed, to ensure $-\frac{\log(\epsilon)}{2\pi} \geq \frac{2}{r_\Gamma}$ and $-\frac{\log(\epsilon)}{2\pi} \geq \frac{2e^t}{r_\Gamma}$. Therefore, we have the lower bounds $y \geq \frac{2}{r_\Gamma}$ and $y \geq \frac{2e^t}{r_\Gamma}$. Now, by equation (2.5.3) we deduce

$$d_{\mathrm{hyp}}(z,\gamma(z)) \geq d_{\mathrm{hyp}}\left(iy, \frac{i}{r_\Gamma^2 y}\right).$$

Moreover, the hypothesis $y \geq \frac{2}{r_\Gamma}$ implies $\frac{1}{r_\Gamma^2 y} \leq \frac{1}{2r_\Gamma}$. Therefore, we obtain

$$d_{\mathrm{hyp}}\left(iy, \frac{i}{r_\Gamma^2 y}\right) \geq d_{\mathrm{hyp}}\left(i\frac{2}{r_\Gamma}, \frac{i}{2r_\Gamma}\right) = \log\left(\frac{2}{r_\Gamma}\right) - \log\left(\frac{1}{2r_\Gamma}\right) = \log(4).$$

We conclude that

$$d_{\mathrm{hyp}}(z,\gamma(z)) \geq \log(4) > 1 \qquad (\gamma\in\Gamma\setminus\Gamma_\infty). \qquad (2.5.10)$$

Writing the sum occurring in (2.5.9) as a Stieltjes integral, we find

$$\sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))) = \int_0^\infty K_k(t,v)\mathrm{dN}_{\Gamma\backslash\Gamma_\infty}(z,v).$$

Applying lemma 2.5.2, yields

$$\sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))) \ll y \int_{a_\Gamma(z)}^\infty K_k(t,v)e^v \mathrm{d}v.$$

By equation (2.5.10) we can apply proposition 2.3.4 with $\delta = 1$. The result is

$$\sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))) \ll_{k,t,\Gamma} y \int_{a_\Gamma(z)}^\infty e^{v-\frac{v^2}{4t}} \mathrm{d}v.$$

Recall that $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right)$, then

$$\int_{a_\Gamma(z)}^\infty e^{v-\frac{v^2}{4t}} \mathrm{d}v = 2\sqrt{t}e^t \int_{a_\Gamma(z)}^\infty e^{-\left(\frac{v}{2\sqrt{t}}-\sqrt{t}\right)^2} \frac{\mathrm{d}v}{2\sqrt{t}} = 2\sqrt{t}e^t \operatorname{erfc}\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \sqrt{t}\right).$$

From the lower bound $y \geq \frac{2e^t}{r_\Gamma}$ we have $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right) > 2t$. Thus we can apply the estimate $\mathrm{erfc}(Z) \leq e^{-Z^2}$ for $Z \geq 0$ for the complementary error function. Then

$$\int\limits_{a_\Gamma(z)}^{\infty} e^{v-\frac{v^2}{4t}}\mathrm{d}v \leq 2\sqrt{t}e^t e^{-\frac{a_\Gamma(z)^2}{4t}+a_\Gamma(z)-t}.$$

Finally, replacing $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right)$, we obtain

$$\int\limits_{a_\Gamma(z)}^{\infty} e^{v-\frac{v^2}{4t}}\mathrm{d}v \leq 2\sqrt{t}\left(\frac{r_\Gamma^2 y^2}{2}\right)^{1-\frac{\log\left(\frac{r_\Gamma^2 y^2}{2}\right)}{4t}}.$$

Since the exponent goes to $-\infty$ for $y\to\infty$, and since the quantity under consideration is a bound for the sum that we wanted to estimate we deduce, noting that throughout the argument we considered t fixed, that

$$\lim_{y\to\infty} \sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t; d_{\mathrm{hyp}}(z,\gamma(z))) = 0.$$

This proves that the integral occurring in (2.5.7) is convergent. Now we address the convergence of the last term (2.5.8). Setting $d(z,n) := d_{\mathrm{hyp}}(z, z+n)$ and using the explicit expression for the hyperbolic volume form, we obtain

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\mu_{\mathrm{hyp}}(z)$$

$$= 2\int\limits_0^1 \int\limits_0^{-\frac{\log(\epsilon)}{2\pi}} \sum_{n=1}^{\infty}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d(z,n))\frac{\mathrm{d}y}{y^2}\mathrm{d}x.$$

We bound the integral by the absolute value of the integrand, and the sum by the absolute value of each summand. Moreover, since $d(z,n) = \operatorname{arccosh}\left(1+\frac{n^2}{2y^2}\right)$ is independent of x, we perform the trivial x-integration. Thus,

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll \int\limits_0^{-\frac{\log(\epsilon)}{2\pi}} \sum_{n=1}^{\infty} K_k(t;d(z,n))\frac{\mathrm{d}y}{y^2}.$$

To apply the standard bound on the heat kernel we need a positive uniform lower bound on the hyperbolic distance. Since $y < -\frac{\log(\epsilon)}{2\pi}$, we have

$$d(z,n) = \operatorname{arccosh}\left(1+\frac{n^2}{2y^2}\right) \geq \operatorname{arccosh}\left(1+\frac{2\pi^2}{(-\log(\epsilon))^2}\right),$$

therefore we can apply proposition 2.3.4 with $\delta = \operatorname{arccosh}\left(1 + \frac{2\pi^2}{(-\log(\epsilon))^2}\right) > 0$ and obtain

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll_{\epsilon,k,t} \int\limits_0^{-\frac{\log(\epsilon)}{2\pi}} \sum_{n=1}^\infty e^{-\frac{d(z,n)^2}{4t}} \frac{\mathrm{d}y}{y^2}.$$

Using the estimate

$$d(z,n) = \operatorname{arccosh}\left(1 + \frac{n^2}{2y^2}\right) = \log\left(\frac{n^2}{2y^2} + 1 + \sqrt{\left(\frac{n^2}{2y^2}+1\right)^2 - 1}\right) \geq \log\left(\frac{n^2}{2y^2}+1\right),$$

and the change of variables $v = \frac{1}{y}$, we have

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll_{\epsilon,k,t} \int\limits_{-\frac{2\pi}{\log(\epsilon)}}^\infty \sum_{n=1}^\infty \left(\frac{n^2v^2}{2}+1\right)^{-\frac{\log\left(\frac{n^2v^2}{2}+1\right)}{4t}} \mathrm{d}v.$$

Let $n_{\epsilon,t} \in \mathbb{N}$ be minimal such that $\log\left(\frac{\left(n_{\epsilon,t}\frac{2\pi}{\log(\epsilon)}+1\right)^2}{2}\right) \geq 4t$. Then

$$\begin{aligned}\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll_{\epsilon,k,t} &\sum_{n=1}^{n_{\epsilon,t}} \int\limits_{-\frac{2\pi}{\log(\epsilon)}}^\infty \left(\frac{n^2v^2}{2}+1\right)^{-\frac{\log\left(\frac{n^2v^2}{2}+1\right)}{4t}} \mathrm{d}v \\ &+ \int\limits_{-\frac{2\pi}{\log(\epsilon)}}^\infty \sum_{n=n_{\epsilon,t}}^\infty \left(\frac{n^2v^2}{2}+1\right)^{-\frac{\log\left(\frac{n^2v^2}{2}+1\right)}{4t}} \mathrm{d}v.\end{aligned}$$

In the first summand, where n has finite range, we use the bound

$$\left(\frac{n^2v^2}{2}+1\right)^{-\frac{\log\left(\frac{n^2v^2}{2}+1\right)}{4t}} \leq \left(\frac{v^2}{2}+1\right)^{-\frac{\log\left(\frac{v^2}{2}+1\right)}{4t}} \qquad (n \in \{1,...,n_{\epsilon,t}\}),$$

which is valid because the exponent is always negative. By construction of $n_{\epsilon,t}$, in the second summand we can apply the bound

$$\left(\frac{n^2v^2}{2}+1\right)^{-\frac{\log\left(\frac{n^2v^2}{2}+1\right)}{4t}} \leq \left(\frac{n^2v^2}{2}+1\right)^{-1} \leq \frac{2}{n^2v^2} \qquad (n \geq n_{\epsilon,t}).$$

We thus obtain

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z)-K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll_{\epsilon,k,t} n_{\epsilon,t} \int\limits_{-\frac{2\pi}{\log(\epsilon)}}^{\infty} \left(\frac{v^2}{2}+1\right)^{-\frac{\log\left(\frac{v^2}{2}+1\right)}{4t}} \mathrm{d}v$$

$$+ \int\limits_{-\frac{2\pi}{\log(\epsilon)}}^{\infty} \sum_{n=n_{\epsilon,t}}^{\infty} \frac{2}{n^2v^2}\mathrm{d}v.$$

The first integral converges because, since t is fixed, for v large enough it holds

$$\frac{\log\left(\frac{v^2}{2}+1\right)}{4t} > 1,$$

and the integrand can be bounded by $\frac{2}{v^2}$. For the second term we observe the equality

$$\int\limits_{-\frac{2\pi}{\log(\epsilon)}}^{\infty} \sum_{n=n_{\epsilon,t}}^{\infty} \frac{2}{n^2v^2}\mathrm{d}v = \left(\sum_{n=n_{\epsilon,t}}^{\infty} \frac{2}{n^2}\right)\left(\int_{-\frac{2\pi}{\log(\epsilon)}}^{\infty} \frac{\mathrm{d}v}{v^2}\right),$$

which is the product of a convergent series with a convergent integral. This implies

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z)-K_k(t;0)\right)\mu_{\mathrm{hyp}}(z) \ll_{\epsilon,k,t} 1,$$

i.e., the convergence of the integral occurring in formula (2.5.8).
We now proved the convergence of the integrals (2.5.6), (2.5.7) and (2.5.8). The equivalence of the two sides of the statement of the proposition follows by an unfolding argument. To simplify the proof we assume, without loss of generality, that there is only the cusp $i\infty$. Moreover, let us adopt the shortening

$$\widetilde{K}_k(t,z,\gamma) = j(\gamma,z)^k \left(\frac{\gamma z-\bar{z}}{z-\gamma\bar{z}}\right)^k K_k(t;d_{\mathrm{hyp}}(z,\gamma(z))).$$

We begin by writing the terms occurring in the statement of the proposition in a convenient form to be compared. The left hand side of the formula is

$$\mathrm{HTr}\, K_k^\Gamma(t) = \int\limits_{\mathcal{F}_\Gamma} \mathrm{H}K_k^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z).$$

The term (2.5.6) is

$$\int\limits_{\mathcal{F}_\Gamma\setminus B_\epsilon(i\infty)} \left(K_k^\Gamma(t;z,z)-K_k(t;0)\right)\,\mu_{\mathrm{hyp}}(z)$$

$$= \int\limits_{\mathcal{F}_\Gamma\setminus B_\epsilon(i\infty)} \mathrm{H}K_k^\Gamma(t;z,z) + \sum_{[\eta]\in\Gamma_\infty\setminus\Gamma}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \widetilde{K}_k(t,z,\eta^{-1}\gamma^n\eta)\,\mu_{\mathrm{hyp}}(z).$$

The term (2.5.7) is

$$\int\limits_{\mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi}<y\right)} \left(K_k^\Gamma(t;z,z) - K_k^\infty(t;z,z)\right)\,\mu_{\mathrm{hyp}}(z)$$

$$= \int_{B_\epsilon(i\infty)} \mathrm{H}K_k^\Gamma(t;z,z) + \sum_{\substack{[\eta]\in\Gamma_\infty\backslash\Gamma \\ [\eta]\neq\Gamma_\infty}}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \widetilde{K}_k(t,z,\eta^{-1}\gamma^n\eta)\,\mu_{\mathrm{hyp}}(z).$$

For the expression of the last term (2.5.8), we unfold the integration domain along the cosets of Γ_∞ in Γ to obtain

$$\int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \left(K_k^\infty(t;z,z) - K_k(t;0)\right)\,\mu_{\mathrm{hyp}}(z)$$

$$= \int\limits_{\mathcal{F}_\Gamma\backslash B_\epsilon(i\infty)} \sum_{[\eta]\in\Gamma_\infty\backslash\Gamma}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \widetilde{K}_k(t,z,\eta^{-1}\gamma\eta)\,\mu_{\mathrm{hyp}}(z) + \int\limits_{B_\epsilon(i\infty)} \sum_{\substack{[\eta]\in\Gamma_\infty\backslash\Gamma \\ [\eta]\neq\Gamma_\infty}}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \widetilde{K}_k(t,z,\eta^{-1}\gamma\eta)\,\mu_{\mathrm{hyp}}(z).$$

It is now transparent, by examining the expressions for (2.5.6) and (2.5.7), that the hyperbolic contribution is integrated over the whole fundamental domain $\mathcal{F}_\Gamma$, therefore matching the contribution of the hyperbolic heat trace. Moreover, the parabolic contributions coming from (2.5.8) cancel the parabolic contribution away from the cusps coming from (2.5.6) and the parabolic contribution at the cusps coming from (2.5.7). $\square$

To complete the trace regularization we define, again following Jorgenson–Lundelius [39], the standard trace. Then, combining results from the last two sections and a computation along the lines of D'Hoker–Phong [17], we provide an explicit expression for it.

Definition 2.5.5. The *standard trace of* $K_k^\Gamma(t;z,w)$ is defined by adding the *identity contribution*

$$\mathrm{ITr}\, K_k^\Gamma(t) := \int\limits_X K_k(t;0)\,\mu_{\mathrm{hyp}}(z)$$

to the hyperbolic heat trace, i.e.,

$$\mathrm{STr}\, K_k^\Gamma(t) := \mathrm{ITr}\, K_k^\Gamma(t) + \mathrm{HTr}\, K_k^\Gamma(t).$$

Observe that the standard trace equals the usual trace if $Y(\Gamma)$ is compact.

Proposition 2.5.6. *The contribution of the identity has the value*[1]

$$\mathrm{ITr}\, K_k^\Gamma(t) = \mathrm{vol}_{\mathrm{hyp}}(X)\left(\sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t\,(2k-j)(j+1)} + e^{-tk(k+1)}K_0(t;0)\right).$$

[1] The corresponding expression in [17] differs from our result by a factor 2.

Proof. Since $K_k(t;0)$ is independent of z, one gets

$$\mathrm{ITr}\, K_k^\Gamma(t) = \mathrm{vol}_{\mathrm{hyp}}(X)\, K_k(t;0).$$

We prove the statement by explicitly computing the difference

$$K_k(t;0) - e^{-tk(k+1)} K_0(t;0).$$

Since $\sqrt{\cosh(u)-1} = \sqrt{2}\sinh(u/2)$ and $T_{2k}(\cosh(u/2)) = \cosh(ku)$, formula (2.2.5) implies

$$K_k(t;0) - e^{-tk(k+1)} K_0(t;0) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{(4\pi t)^{\frac{3}{2}}} \int_0^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sinh(u/2)} \left(\cosh(ku) - \cosh(0)\right) \mathrm{d}u.$$

Then, expanding $\cosh(ku)$ and $\sinh(u/2)$ in terms of exponential, we find the relation

$$\frac{\cosh(ku)-1}{\sinh(u/2)} = \sum_{j=0}^{k-1} \left(e^{u\left(k-j-\frac{1}{2}\right)} - e^{-u\left(k-j-\frac{1}{2}\right)} \right).$$

It follows that

$$K_k(t;0) - e^{-tk(k+1)} K_0(t;0) = \sum_{j=0}^{k-1} \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{(4\pi t)^{\frac{3}{2}}} \int_0^\infty ue^{-\frac{u^2}{4t}} \left(e^{u\left(k-j-\frac{1}{2}\right)} - e^{-u\left(k-j-\frac{1}{2}\right)} \right) \mathrm{d}u.$$

We now integrate each of these summands. Let b be a half-integer, and observe the relations

$$-\frac{u^2}{4t} \pm bu = -\left(\frac{u}{2\sqrt{t}} \mp \sqrt{t}b\right)^2 + tb^2.$$

Then, we obtain

$$\begin{aligned}
&\int_0^\infty ue^{-\frac{u^2}{4t}+bu}\mathrm{d}u - \int_0^\infty ue^{-\frac{u^2}{4t}-bu}\mathrm{d}u \\
&\qquad = 4te^{tb^2}\left(\int_0^\infty \frac{u}{2\sqrt{t}} e^{-\left(\frac{u}{2\sqrt{t}}-\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} - \int_0^\infty \frac{u}{2\sqrt{t}} e^{-\left(\frac{u}{2\sqrt{t}}+\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}}\right) \\
&\qquad = 4te^{tb^2}\Bigg(\int_0^\infty \left(\frac{u}{2\sqrt{t}} - \sqrt{t}b\right) e^{-\left(\frac{u}{2\sqrt{t}}-\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} + \sqrt{t}b \int_0^\infty e^{-\left(\frac{u}{2\sqrt{t}}-\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} \\
&\qquad\quad - \int_0^\infty \left(\frac{u}{2\sqrt{t}} + \sqrt{t}b\right) e^{-\left(\frac{u}{2\sqrt{t}}+\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} + \sqrt{t}b \int_0^\infty e^{-\left(\frac{u}{2\sqrt{t}}+\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} \Bigg).
\end{aligned}$$

The first and third integrals can be explicitly computed through the changes of variables $v = \frac{u}{2\sqrt{t}} - \sqrt{t}b$ and $v = \frac{u}{2\sqrt{t}} + \sqrt{t}b$, respectively. The results are

$$\int_0^\infty \left(\frac{u}{2\sqrt{t}} - \sqrt{t}b\right) e^{-\left(\frac{u}{2\sqrt{t}}-\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} = \int_{-\sqrt{t}b}^\infty ve^{-v^2}\mathrm{d}v = \frac{e^{-tb^2}}{2},$$

and

$$\int_0^\infty \left(\frac{u}{2\sqrt{t}} + \sqrt{t}b\right) e^{-\left(\frac{u}{2\sqrt{t}}+\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} = \int_{\sqrt{t}b}^\infty v e^{-v^2}\mathrm{d}v = \frac{e^{-tb^2}}{2}.$$

Since they appear with opposite sign, they cancel each other. Also for the remaining two integrals we can apply the changes of variables $v = \frac{u}{2\sqrt{t}} - \sqrt{t}b$ and $v = \frac{u}{2\sqrt{t}} + \sqrt{t}b$, respectively. We obtain

$$\sqrt{t}b \int_0^\infty e^{-\left(\frac{u}{2\sqrt{t}}-\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} = \sqrt{t}b \int_{-\sqrt{t}b}^\infty e^{-v^2}\mathrm{d}v,$$

and

$$\sqrt{t}b \int_0^\infty e^{-\left(\frac{u}{2\sqrt{t}}+\sqrt{t}b\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} = \sqrt{t}b \int_{\sqrt{t}b}^\infty e^{-v^2}\mathrm{d}v.$$

Observe that the integrands in the last two expressions are even, and the lower limits of the integrals are symmetric across the origin. Thus, using the integration formula for the Gaussian integral $\int_0^\infty e^{-v^2}\mathrm{d}v = \frac{\sqrt{\pi}}{2}$, we have

$$\sqrt{t}b \int_{-\sqrt{t}b}^\infty e^{-v^2}\mathrm{d}v + \sqrt{t}b \int_{\sqrt{t}b}^\infty e^{-v^2}\mathrm{d}v = 2\sqrt{t}b \int_0^\infty e^{-v^2}\mathrm{d}v = \sqrt{\pi t}\, b.$$

Summing up, we have computed

$$\int_0^\infty u e^{-\frac{u^2}{4t}+bu}\mathrm{d}u - \int_0^\infty u e^{-\frac{u^2}{4t}-bu}\mathrm{d}u = 4\sqrt{\pi} b t^{\frac{3}{2}} e^{tb^2} \qquad (2b \in \mathbb{Z})\,.$$

Using this expression with $b = k - j - \frac{1}{2}$ yields

$$\begin{aligned} K_k(t;0) - e^{-tk(k+1)}K_0(t;0) &= \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t\left(k+\frac{1}{2}\right)^2 + t\left(k-j-\frac{1}{2}\right)^2} \\ &= \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t\,(2k-j)(j+1)}. \end{aligned} \tag{2.5.11}$$

Which is equivalent to the statement to be proven. □

Summing up propositions 2.4.4, 2.4.5 and 2.5.6, we deduce the decomposition

$$\operatorname{STr} K_k^\Gamma(t) = \mathrm{vol}_{\mathrm{hyp}}(X) \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t\,(2k-j)(j+1)} \tag{2.5.12}$$

$$+ \mathrm{vol}_{\mathrm{hyp}}(X) e^{-tk(k+1)} K_0(t;0) \tag{2.5.13}$$

$$+ \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}}. \tag{2.5.14}$$

2.6 Definition of the regularized determinant

In this section we define and prove the well-definedness of the regularization of the determinant of the Laplacian Δ_k. This regularized determinant of Δ_k is obtained by replacing the usual heat trace by the standard trace in definition 1.4.15.

Definition 2.6.1. The *number of zero modes* of $\overline{M}_{-k}$ is

$$N_k := \lim_{t \to \infty} \operatorname{STr} K_k^\Gamma(t).$$

The *regularized spectral ζ-function associated to* Δ_k is given by the formula

$$\zeta_k^\Gamma(s) := \frac{1}{\Gamma(s)} \mathcal{M}\left(\operatorname{STr} K_k^\Gamma(t) - N_k, s\right).$$

Moreover, the *regularized determinan of* Δ_k is

$$\det\nolimits_\Gamma^*(\Delta_k) := \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^\Gamma(s)\right)_{s=0}.$$

The lack of the superscript Γ in the definition of the zero modes is justified by the upcoming lemma 2.6.3, which shows that N_k is independent of Γ. To ensure the well-definedness of the regularized determinant we have to prove that the regularized spectral ζ-function is holomorphic at $s = 0$. Before proving it, we need two preliminary results.

Lemma 2.6.2. *There exists a constant $b_\Gamma \in \mathbb{R}_{>0}$, only dependent on Γ, such that, for $t \to 0$, we have the asymptotic expansions*

$$\operatorname{ITr} K_k^\Gamma(t) = \frac{\mathrm{vol}_{\mathrm{hyp}}(X)}{4\pi t} - \frac{\mathrm{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi} + O_{\Gamma,k}(t),$$
$$\operatorname{HTr} K_k^\Gamma(t) = O_{\Gamma,k}\left(e^{-\frac{b_\Gamma}{t}}\right).$$

Proof. Regarding the behavior of the identity contribution, we have to show

$$K_k(t;0) = \frac{1}{4\pi t} - \frac{3k+1}{12\pi} + O_k(t) \qquad (t \to 0). \tag{2.6.1}$$

Given the auxiliary function

$$f_k(t) = (4\pi t)\,K_k(t;0) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}} \int_0^\infty \frac{u e^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \cosh(ku)\mathrm{d}u,$$

it is enough to compute the limits of $f_k(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t} f_k(t)$ for $t \to 0$. We preliminary state the values of the Gaussian integral and of two of its variations, computed integrating by parts, which will be used in the proof. They are

$$\int_0^\infty e^{-v^2} \mathrm{d}v = \frac{\sqrt{\pi}}{2}, \qquad \int_0^\infty v^2 e^{-v^2} \mathrm{d}v = \frac{\sqrt{\pi}}{4}, \qquad \int_0^\infty v^4 e^{-v^2} \mathrm{d}v = \frac{3\sqrt{\pi}}{8}. \tag{2.6.2}$$

Applying the change of variables $v = \frac{u}{2\sqrt{t}}$, we find

$$\lim_{t \to 0} f_k(t) = \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{t} v e^{-v^2}}{\sinh\left(\sqrt{t} v\right)} \cosh(2k\sqrt{t} v) \mathrm{d}v.$$

Since the limit

$$\lim_{t \to 0} \left(e^{-t\left(k+\frac{1}{2}\right)^2} \frac{\sqrt{t} v e^{-v^2}}{\sinh\left(\sqrt{t} v\right)} \cosh(2k\sqrt{t} v) \right) = e^{-v^2}$$

is uniform, we can exchange limit and integral to obtain

$$\lim_{t \to 0} f_k(t) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-v^2} \mathrm{d}v = 1,$$

where we used formula (2.6.2) for the value of the integral. We now compute the limit for the first derivative of $f_k(t)$. We carry the t-factors into the integral and expand the derivative, observing that we can differentiate under the integral sign by Lebesgue's theorem. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} (f_k(t)) = &- \left(k + \frac{1}{2}\right)^2 f_k(t) - \int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\sqrt{\pi} t^{\frac{3}{2}}} \frac{u e^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \cosh(ku) \mathrm{d}u \\ &+ \int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{8\sqrt{\pi} t^{\frac{5}{2}}} \frac{u^3 e^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \cosh(ku) \mathrm{d}u. \end{aligned}$$

We change variables to $v = \frac{u}{2\sqrt{t}}$, and obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} (f_k(t)) = &- \left(k + \frac{1}{2}\right)^2 f_k(t) - \int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi} t} \frac{\sqrt{t} v \cosh(2k\sqrt{t} v)}{\sinh(\sqrt{t} v)} e^{-v^2} \mathrm{d}v \\ &+ \int_0^\infty \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi} t} \frac{\sqrt{t} v \cosh(2k\sqrt{t} v)}{\sinh(\sqrt{t} v)} v^2 e^{-v^2} \mathrm{d}v. \end{aligned}$$

The Taylor series of $\cosh(2k\sqrt{t} v)$ and $\sinh(\sqrt{t} v)$ are given by

$$\cosh(2k\sqrt{t} v) = 1 + \frac{4k^2 t v^2}{2} + O((2k\sqrt{t} v)^4),$$

and

$$\sinh(\sqrt{t}v) = \sqrt{t}v + \frac{t^{\frac{3}{2}}v^3}{6} + O((\sqrt{t}v)^5),$$

respectively. Their combination yields

$$\frac{\sqrt{t}v\,\cosh(2k\sqrt{t}v)}{\sinh(\sqrt{t}v)} = 1 + \left(2k^2 - \frac{1}{6}\right) tv^2 + O(t^2v^4).$$

Using the Taylor series computed above we have

$$\begin{aligned}
-\int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \frac{\sqrt{t}v\,\cosh(2k\sqrt{t}v)}{\sinh(\sqrt{t}v)} e^{-v^2}\mathrm{d}v + \int_0^\infty \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \frac{\sqrt{t}v\,\cosh(2k\sqrt{t}v)}{\sinh(\sqrt{t}v)} v^2e^{-v^2}\mathrm{d}v \\
= -\int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(1 + \left(2k^2 - \frac{1}{6}\right) tv^2 + O_k(t^2v^4)\right) e^{-v^2}\mathrm{d}v \\
+ \int_0^\infty \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(1 + \left(2k^2 - \frac{1}{6}\right) tv^2 + O_k(t^2v^4)\right) v^2e^{-v^2}\mathrm{d}v.
\end{aligned}$$

Applying formula (2.6.2), we compute

$$\int_0^\infty \left(\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} e^{-v^2} - \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} v^2e^{-v^2}\right) \mathrm{d}v = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(\frac{\sqrt{\pi}}{2} - 2\frac{\sqrt{\pi}}{4}\right) = 0.$$

Thus, we can simplify the singular terms in the Taylor expansions occurring in the integrals and take the t-limits in the u-integration. This yields

$$\begin{aligned}
-\lim_{t\to 0}\int_0^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(\frac{\sqrt{t}v\,\cosh(2k\sqrt{t}v)}{\sinh(\sqrt{t}v)} - 1\right) e^{-v^2}\mathrm{d}v \\
= -\int_0^\infty \lim_{t\to 0} \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(\left(2k^2 - \frac{1}{6}\right) tv^2 + O_k(t^2v^4)\right) e^{-v^2}\mathrm{d}v \\
= -\int_0^\infty \frac{1}{\sqrt{\pi}} \left(2k^2 - \frac{1}{6}\right) v^2e^{-v^2}\mathrm{d}v,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t\to 0}\int_0^\infty \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(\frac{\sqrt{t}v\,\cosh(2k\sqrt{t}v)}{\sinh(\sqrt{t}v)} - 1\right) v^2e^{-v^2}\mathrm{d}v \\
= \int_0^\infty \lim_{t\to 0} \frac{2e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{\pi t}} \left(\left(2k^2 - \frac{1}{6}\right) tv^2 + O_k(t^2v^4)\right) v^2e^{-v^2}\mathrm{d}v \\
= \int_0^\infty \frac{2}{\sqrt{\pi}} \left(2k^2 - \frac{1}{6}\right) v^4e^{-v^2}\mathrm{d}v.
\end{aligned}$$

Using the values stated in formula (2.6.2), we have

$$-\int_0^\infty \frac{1}{\sqrt{\pi}}\left(2k^2-\frac{1}{6}\right)v^2e^{-v^2}\mathrm{d}v=-\left(2k^2-\frac{1}{6}\right)\frac{1}{4},$$

and

$$\int_0^\infty \frac{2}{\sqrt{\pi}}\left(2k^2-\frac{1}{6}\right)v^4e^{-v^2}\mathrm{d}v=\left(2k^2-\frac{1}{6}\right)\frac{3}{4}.$$

Summing up the contributions yield

$$\lim_{t\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\left(f_k(t)\right)=-\left(k+\frac{1}{2}\right)^2-\left(2k^2-\frac{1}{6}\right)\frac{1}{4}+\left(2k^2-\frac{1}{6}\right)\frac{3}{4}=-\left(k+\frac{1}{3}\right).$$

Which completes the proof of formula (2.6.1) and of the asymptotic expansion of the identity trace.
It remains to estimate the contribution of the hyperbolic heat trace. An application of propositions 2.4.4 and 2.4.5 gives the expression

$$\mathrm{HTr}\,K_k^\Gamma(t)=\sum_{\gamma\in H(\Gamma)}\sum_{n=1}^\infty\frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}\frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}}.$$

As already stated in equation (2.4.4), the lengths of closed geodesics $\ell(\gamma)$ are all positively uniformly bounded from below: Let $c_\Gamma\in\mathbb{R}_{>0}$ be such a lower bound. Then

$$\mathrm{HTr}\,K_k^\Gamma(t)=\frac{e^{-\frac{c_\Gamma^2}{4t}}}{\sqrt{t}}\sum_{\gamma\in H(\Gamma)}\sum_{n=1}^\infty\frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}\frac{e^{-t\left(k+\frac{1}{2}\right)^2-\frac{n^2\,\ell(\gamma)^2-c_\Gamma^2}{4t}}}{4\sqrt{\pi}}\ll_{\Gamma,k}\frac{e^{-\frac{c_\Gamma^2}{4t}}}{\sqrt{t}},$$

because each summand decreases monotonically for $t\to 0$. Thus, for some $b_\Gamma\in\mathbb{R}_{>0}$ we have

$$\mathrm{HTr}\,K_k^\Gamma(t)=O_{\Gamma,k}\big(e^{-\frac{b_\Gamma}{t}}\big)\qquad(t\to 0)\,.$$

This completes the proof of the lemma. □

Now we turn our attention to the asymptotic of the standard trace for t large.

Lemma 2.6.3. *Let $d_{\Gamma,k}=\min\left\{\frac{1}{4},\lambda_{1,\Gamma,k}\right\}$, where $\lambda_{1,\Gamma,k}$ is the first non-zero eigenvalue of Δ_k. Then, for $t\to\infty$, we have*

$$\mathrm{ITr}\,K_k^\Gamma(t)=O_{\Gamma,k}\big(e^{-\frac{t}{4}}\big),$$
$$\mathrm{HTr}\,K_k^\Gamma(t)=\begin{cases}1+O_\Gamma\big(e^{-td_{\Gamma,0}}\big), & k=0,\\ O_{\Gamma,k}\big(e^{-td_{\Gamma,k}}\big), & k\geq 1.\end{cases}$$

In particular $N_0=1$ and $N_k=0$ for $k\geq 1$.

Proof. We begin by examining the large time behavior of $K_0(t;0)$. Applying formula (2.2.5) we obtain

$$K_0(t;0) \approx \frac{e^{-\frac{t}{4}}}{t^{\frac{3}{2}}} \int_0^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \mathrm{d}u.$$

Then we observe that, since $\frac{u}{\sinh(u/2)} \approx 1$ for $0 \le u \le 2$ and $\sinh(u/2) > 1$ for $u > 2$, we have

$$\begin{aligned}\int_0^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \mathrm{d}u &= \int_0^2 \frac{ue^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \mathrm{d}u + \int_2^\infty \frac{ue^{-\frac{u^2}{4t}}}{\sinh\left(\frac{u}{2}\right)} \mathrm{d}u \approx \int_0^2 e^{-\frac{u^2}{4t}} \mathrm{d}u + \int_2^\infty ue^{-\frac{u^2}{4t}} \mathrm{d}u \\ &\approx 2\sqrt{t} \int_0^{\frac{1}{\sqrt{t}}} e^{-v^2} \mathrm{d}v + 4t \int_{\frac{1}{\sqrt{t}}}^\infty ve^{-v^2} \mathrm{d}v.\end{aligned}$$

Both the integrals in the last expression are finite for $t \to \infty$, therefore we have

$$K_0(t;0) \ll \frac{e^{-\frac{t}{4}}}{t^{\frac{3}{2}}} \left(2\sqrt{t} + 4t\right) \ll \frac{e^{-\frac{t}{4}}}{t} + \frac{e^{-\frac{t}{4}}}{\sqrt{t}} = O\big(e^{-\frac{t}{4}}\big) \qquad (t \to \infty).$$

Let us observe that $(2k - j)(j + 1) \ge 2k$. Indeed, if $j = 0$ the equality holds, and if $j > 0$ expanding the product and simplifying the terms the claimed inequality reduces to the inequality $2k > j + 1$, which is satisfied for any $j \in \{1, ..., k - 1\}$. Therefore we have

$$\sum_{j=0}^{k-1} \frac{2k - 2j - 1}{4\pi} e^{-t\,(2k-j)(j+1)} = O_k\left(e^{-2tk}\right) \qquad (t \to \infty).$$

Further noting that $k(k + 1) \ge 2k$ for $k \ge 1$, the last estimate and formula (2.5.11) imply

$$K_k(t;0) = O\left(e^{-t\,\max\left\{2k, \frac{1}{4}\right\}}\right) \qquad (t \to \infty). \tag{2.6.3}$$

This completes the claimed expansion of the identity trace. We now examine the contribution of the hyperbolic heat trace, which, by propositions 2.4.4 and 2.4.5, is given by the formula

$$\mathrm{HTr}\, K_k^\Gamma(t) = \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^\infty \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}}.$$

We begin with the case $k = 0$. By the Selberg trace formula for non-compact Riemann surfaces [34, chapter 8, theorem 4.2] applied to the test functions

$$h(r) = e^{-t\left(r^2+\frac{1}{4}\right)}, \qquad g(u) = \hat{h}(u) = \frac{e^{-\frac{t}{4} - \frac{u^2}{4t}}}{\sqrt{4\pi t}},$$

and to the trivial character, we have

$$\mathrm{HTr}\, K_k^{\Gamma}(t) = \sum_{n=0}^{\infty} e^{-t\lambda_{n,\Gamma,0}} - \frac{\mathrm{vol}_{\mathrm{hyp}}(X)}{4\pi} \int\limits_{-\infty}^{\infty} re^{-t\left(r^2+\frac{1}{4}\right)} \tanh(\pi r)\mathrm{d}r - \frac{e^{-\frac{t}{4}}}{4} \operatorname{Tr}\left(\mathrm{I} - \Phi\left(\frac{1}{2}\right)\right)$$
$$- \frac{1}{4\pi} \int\limits_{-\infty}^{\infty} e^{-t\left(r^2+\frac{1}{4}\right)} \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)}\mathrm{d}r + \frac{p\, e^{-\frac{t}{4}}}{\sqrt{4\pi t}} \log(2) + \frac{p}{2\pi} \int\limits_{-\infty}^{\infty} e^{-t\left(r^2+\frac{1}{4}\right)} \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\mathrm{d}r.$$

In this formula $\{\lambda_{n,\Gamma,0}\}_{n=0}^{\infty}$ is the discrete spectrum of $D_0 = \Delta_0$, $\Phi(s)$ is the scattering matrix associated to the cusps, and $\varphi(s) = \det \Phi(s)$. Moreover, we note that we ignored the two terms of the cited formula corresponding to the elliptic points and to the choice of the character, respectively, because in our setting they are identically zero. Now, considering the limit for $t\to\infty$, for the first term we have

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-t\lambda_{n,\Gamma,0}} &= 1 + e^{-t\lambda_{1,\Gamma,0}} \sum_{n=1}^{\infty} e^{-t(\lambda_{n,\Gamma,0}-\lambda_{1,\Gamma,0})} \\ &\leq 1 + e^{-t\lambda_{1,\Gamma,0}} \sum_{n=1}^{\infty} e^{-(\lambda_{n,\Gamma,0}-\lambda_{1,\Gamma,0})} = 1 + O_{\Gamma}\left(e^{-t\lambda_{1,\Gamma,0}}\right),\end{aligned}$$

for the second we have

$$-\frac{\mathrm{vol}_{\mathrm{hyp}}(X)}{4\pi} \int\limits_{-\infty}^{\infty} re^{-t\left(r^2+\frac{1}{4}\right)} \tanh(\pi r)\mathrm{d}r \ll -e^{-\frac{t}{4}} \frac{\mathrm{vol}_{\mathrm{hyp}}(X)}{4\pi} \int\limits_{-\infty}^{\infty} re^{-r^2} \tanh(\pi r)\mathrm{d}r = O_{\Gamma}\big(e^{-\frac{t}{4}}\big),$$

for the third we have

$$-\frac{e^{-\frac{t}{4}}}{4} \operatorname{Tr}\left(\mathrm{I} - \Phi\left(\frac{1}{2}\right)\right) = O_{\Gamma}\big(e^{-\frac{t}{4}}\big),$$

for the fourth we have

$$-\frac{1}{4\pi} \int\limits_{-\infty}^{\infty} e^{-t\left(r^2+\frac{1}{4}\right)} \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)}\mathrm{d}r \ll -\frac{e^{-\frac{t}{4}}}{4\pi} \int\limits_{-\infty}^{\infty} e^{-r^2} \frac{\varphi'\left(\frac{1}{2}+ir\right)}{\varphi\left(\frac{1}{2}+ir\right)}\mathrm{d}r = O_{\Gamma}\big(e^{-\frac{t}{4}}\big),$$

for the fifth we have

$$\frac{p\, e^{-\frac{t}{4}}}{\sqrt{4\pi t}} \log(2) = O_{\Gamma}\big(e^{-\frac{t}{4}}\big),$$

and for the last one we have

$$\frac{p}{2\pi} \int\limits_{-\infty}^{\infty} e^{-t\left(r^2+\frac{1}{4}\right)} \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\mathrm{d}r \ll e^{-\frac{t}{4}} \frac{p}{2\pi} \int\limits_{-\infty}^{\infty} e^{-r^2} \frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\mathrm{d}r = O_{\Gamma}\big(e^{-\frac{t}{4}}\big).$$

This completes the proof for the case $k = 0$. To obtain the statement for $k \geq 1$, it is enough to observe that the hyperbolic heat trace for $k \geq 1$ is related to hyperbolic heat trace for $k = 0$ by a multiplicative factor $e^{-tk(k+1)}$. $\square$

We can now prove the claimed holomorphicity of the spectral ζ-function at $s = 0$.

Proposition 2.6.4. *The function $\zeta_k^\Gamma(s)$ is holomorphic at $s = 0$.*

Proof. A first application of lemmata 2.6.2 and 2.6.3 implies that

$$\mathrm{STr}\, K_k^\Gamma(t) - N_k = \begin{cases} O_{k,\Gamma}\left(\frac{1}{t}\right), & t\to 0, \\ O_{k,\Gamma}(e^{-d_{\Gamma,k}t}), & t\to\infty, \end{cases}$$

therefore

$$\int_0^\infty \left(\mathrm{STr}\, K_k^\Gamma(t) - N_k\right) t^{s-1} \mathrm{d}t$$

is holomorphic for $\mathrm{Re}(s) > 1$. Then lemma 2.6.2 and the direct mapping theorem B.2 for the Mellin transform imply the analytic continuation to $\mathrm{Re}(s) > -1$, with singular expansion at $s = 0$ of the form

$$\mathcal{M}\left(\mathrm{STr}\, K_k^\Gamma(t) - N_k, s\right) = -\frac{\mathrm{vol_{hyp}}(X)\,(3k+1)}{12\pi s} - \frac{N_k}{s} + O_{\Gamma,k}(1) \qquad (s\to 0). \tag{2.6.4}$$

Since $\frac{1}{\Gamma(s)}$ has a simple zero at $s = 0$, the desired holomorphicity of $\zeta_k^\Gamma(s)$ is proven. □

Definition 2.6.5. We define

$$\det{}^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right) := \det{}^*_\Gamma\left(\Delta^0_{\overline{M}_{-k}}\right) := 4^{-\zeta_k^\Gamma(0)} \det{}^*_\Gamma\left(\Delta_k\right),$$

where the first equality is justified by remark 1.4.19, and the second one is justified by observation 1.4.17 and equation (2.2.1).

Observation 2.6.6. Directly by equation (2.6.4) we have

$$\zeta_k^\Gamma(0) = -\frac{\mathrm{vol_{hyp}}(X)\,(3k+1)}{12\pi} - N_k = -\frac{(2g-2+p)\,(3k+1)}{6} - N_k,$$

where the second equality follows from the Gauss–Bonnet theorem.

We remark that proposition 2.6.4 ensures that $\det^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)$ is well-defined. Its explicit value will be given in theorem 2.8.4.

2.7 Computation of the hyperbolic contribution to the regularized determinant

Given the well-definedness of the regularized determinant, the goal for this and the next section is to explicitly compute it. In this section we adapt a computation of D'Hoker–Phong [17, 18] to the non-compact case to provide a formula for the hyperbolic contribution.

By the asymptotic expansions for $\operatorname{ITr} K_k^\Gamma(t)$ for $t\to 0$ and $t\to\infty$ given in lemmata 2.6.2 and 2.6.3, respectively, the function

$$\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right) + \frac{\mathrm{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi s}$$

is holomorphic at $s=0$. Therefore, the quantity

$$c_k := \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \left(\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right) + \frac{\mathrm{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi s} \right)_{s=0} \tag{2.7.1}$$

is finite. We observe that it is independent of Γ.

Definition 2.7.1. The Selberg $\mathcal{Z}$-function associated to Γ is the function defined by the formula

$$\mathcal{Z}_\Gamma(s) = \prod_{\gamma\in H(\Gamma)} \prod_{n=0}^{\infty} \left(1 - e^{-\ell(\gamma)(s+n)}\right)$$

for $s\in\mathbb{C}$ such that $\mathrm{Re}(s) > 1$, and extended by analytic continuation on the whole complex plane.

Proposition 2.7.2. *The derivative of the spectral zeta function has the special value*

$$\left(\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^\Gamma(s)\right)_{s=0} = \begin{cases} -\log\left(\mathcal{Z}_\Gamma'(1)\right) + c_0\,\mathrm{vol}_{\mathrm{hyp}}(X), & k=0,\\ -\log\left(\mathcal{Z}_\Gamma(k+1)\right) + c_k\,\mathrm{vol}_{\mathrm{hyp}}(X), & k\geq 1. \end{cases}$$

Here $\mathcal{Z}_\Gamma(s)$ denotes the Selberg zeta function associated to Γ.

Proof. This computation has been done in [17] separating the cases $k=0$ and $k\geq 1$. To unify the treatment of the two cases we observe that, first by remark B.4, and then by the asymptotics given in lemmata 2.6.2 and 2.6.3, we have

$$\mathcal{M}\left(\operatorname{HTr} K_k^\Gamma(t) - N_k, s\right) = \mathcal{M}\left(\operatorname{HTr} K_k^\Gamma(t), s\right) = \begin{cases} \int_0^\infty \operatorname{HTr} K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t, & \mathrm{Re}(s)<0,\\ \int_0^\infty \left(\operatorname{HTr} K_k^\Gamma(t) - N_k\right) t^{s-1}\mathrm{d}t, & \mathrm{Re}(s)>0. \end{cases}$$

We express the derivative as a limit. Since $\zeta_k^\Gamma(s)$ is holomorphic at $s=0$, the direction we use to go to zero is negligible; we use $s\in\mathbb{R}_{<0}$ for convenience. Then,

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}s}\zeta_k^\Gamma(s)\right)_{s=0} &= \lim_{s\to 0}\frac{\zeta_k^\Gamma(s) - \zeta_k^\Gamma(0)}{s}\\ &= \lim_{s\to 0^-}\left(\frac{1}{s\Gamma(s)}\int_0^\infty \operatorname{HTr} K_k^\Gamma(t) t^{s-1}\mathrm{d}t - \frac{\mathrm{vol}_{\mathrm{hyp}}(X)\,(3k+1) + 12\pi\zeta_k^\Gamma(0)}{12\pi s}\right)\\ &\quad + \lim_{s\to 0^-}\left(\frac{1}{s\Gamma(s)}\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right) + \frac{\mathrm{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi s}\right).\end{aligned}$$

The second limit equals $c_k\,\mathrm{vol}_{\mathrm{hyp}}(X)$ by construction, thus our aim is to compute the first one. Using the special value of the spectral ζ-function of observation (2.6.6), we rewrite it as

$$\lim_{s\to 0^-}\left(\frac{1}{s\Gamma(s)}\int_0^\infty \operatorname{HTr} K_k^\Gamma(t) t^{s-1}\mathrm{d}t + \frac{N_k}{s}\right).$$

We start by manipulating the integral term occurring in the last expression. Applying the change of variables $u = v(v+a)t$ to the integral representation for the Γ-function

$$\Gamma(1-s) = \int_0^\infty u^{-s} e^{-u} \mathrm{d}u \qquad (\mathrm{Re}(s) < 1)\,,$$

and rearranging terms, we obtain the formula

$$t^{s-1} = \frac{1}{\Gamma(1-s)} \int_0^\infty (v(v+a))^{-s} e^{-v(v+a)t}(2v+a)\,\mathrm{d}v \qquad (\mathrm{Re}(s) < 1,\, a \in \mathbb{R}_{>0})\,.$$

We explicit the expression of the Mellin transform of $\mathrm{HTr}\, K_k^\Gamma(t)$ using propositions 2.4.4 and 2.4.5, then we apply the formula above with $a = 2k+1$. We obtain,

$$\begin{aligned}
\frac{1}{s\Gamma(s)} \int_0^\infty \mathrm{HTr}\, K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t =& \frac{1}{s\Gamma(s)} \int_0^\infty \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^\infty \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}} t^{s-1}\mathrm{d}t \\
=& \frac{1}{s\Gamma(s)} \int_0^\infty \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^\infty \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \frac{e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}} \frac{1}{\Gamma(1-s)} \\
& \times \int_0^\infty (v(v+a))^{-s} e^{-v(v+a)t}(2v+a)\,\mathrm{d}v\,\mathrm{d}t.
\end{aligned}$$

Moreover, by Fubini's theorem, we exchange integrals and summations to get

$$\begin{aligned}
\frac{1}{s\Gamma(s)} \int_0^\infty \mathrm{HTr}\, K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t =& \frac{1}{s\Gamma(s)\Gamma(1-s)} \int_0^\infty \frac{2v+2k+1}{(v(v+2k+1))^s} \sum_{\gamma \in H(\Gamma)} \sum_{n=1}^\infty \frac{\ell(\gamma)}{\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)} \\
& \times \int_0^\infty \frac{e^{-t\left(\left(k+\frac{1}{2}\right)^2 + v(v+2k+1)\right) - \frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}}\,\mathrm{d}t\,\mathrm{d}v.
\end{aligned}$$

The t-integral can be explicitly computed by means of Bessel functions. Formula 10.32.10 of [48] states

$$K_\nu(Z) = \frac{1}{2}\left(\frac{Z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{Z^2}{4t}} \frac{dt}{t^{\nu+1}} \qquad \left(|\arg(Z)| < \frac{\pi}{4}\right), \tag{2.7.2}$$

and formula 10.39.2 of loc. cit. states

$$K_{\pm\frac{1}{2}}(Z) = \sqrt{\frac{\pi}{2Z}} e^{-Z}. \tag{2.7.3}$$

Applying them, we deduce

$$\int_0^\infty \frac{e^{-t\left(\left(k+\frac{1}{2}\right)^2+v(v+2k+1)\right)-\frac{n^2\,\ell(\gamma)^2}{4t}}}{4\sqrt{\pi t}}\,\mathrm{d}t$$

$$= \frac{1}{2\sqrt{\pi}}\left(\frac{n\,\ell(\gamma)\,\left(v+k+\frac{1}{2}\right)}{2}\right)^{\frac{1}{2}}\frac{K_{-\frac{1}{2}}\left(n\,\ell(\gamma)\left(v+k+\frac{1}{2}\right)\right)}{v+k+\frac{1}{2}} = \frac{e^{-n\,\ell(\gamma)\,\left(v+k+\frac{1}{2}\right)}}{4\left(v+k+\frac{1}{2}\right)}. \quad (2.7.4)$$

Replacing this computation in the expression we have to evaluate, we have

$$\frac{1}{s\Gamma(s)}\int_0^\infty \mathrm{HTr}\,K_k^\Gamma(t)\,t^{s-1}\mathrm{d}t$$

$$= \frac{1}{s\Gamma(s)\Gamma(1-s)}\int_0^\infty (v(v+2k+1))^{-s}\sum_{\gamma\in H(\Gamma)}\sum_{n=1}^\infty \frac{\ell(\gamma)e^{-n\,\ell(\gamma)\,\left(v+k+\frac{1}{2}\right)}}{2\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}\mathrm{d}v.$$

We now relate the double sum occurring in the last integral to the Selberg $\mathcal{Z}$-function. Since the formula for the derivative of a convergent infinite product is

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\prod_{j=0}^\infty g_j(v)\right) = \left(\sum_{j=0}^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}v}g_j(v)}{g_j(v)}\right)\prod_{v=0}^\infty g_j(v),$$

directly from the definition 2.7.1 of the Selberg zeta function we compute

$$\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1) = \left(\sum_{\gamma\in H(\Gamma)}\sum_{j=0}^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}v}\left(1-e^{-\ell(\gamma)(v+k+1+j)}\right)}{1-e^{-\ell(\gamma)(v+k+1+j)}}\right)\mathcal{Z}_\Gamma(v+k+1).$$

Therefore,

$$\frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} = \sum_{\gamma\in H(\Gamma)}\sum_{j=0}^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}v}\left(1-e^{-\ell(\gamma)(v+k+1+j)}\right)}{1-e^{-\ell(\gamma)(v+k+1+j)}} = \sum_{\gamma\in H(\Gamma)}\sum_{j=0}^\infty \frac{\ell(\gamma)e^{-\ell(\gamma)(v+k+1+j)}}{1-e^{-\ell(\gamma)(v+k+1+j)}}.$$

Now, applying the summation formula for the geometric series, one gets

$$\begin{aligned}\sum_{j=0}^\infty \frac{e^{-\ell(\gamma)(v+k+1+j)}}{1-e^{-\ell(\gamma)(v+k+1+j)}} = \sum_{j=0}^\infty\sum_{n=1}^\infty e^{-\ell(\gamma)(v+k+1+j)\,n} &= \sum_{n=1}^\infty\sum_{j=0}^\infty e^{-\ell(\gamma)(v+k+1)\,n-\ell(\gamma)\,j\,n}\\ &= \sum_{n=1}^\infty \frac{e^{-n\,\ell(\gamma)(v+k+1)}}{1-e^{-n\,\ell(\gamma)}}.\end{aligned}$$

First substituting this expression in the formula for the logarithmic derivative of the Selberg $\mathcal{Z}$-function, and then multiplying numerators and denominators of the summands by $e^{\frac{n\,\ell(\gamma)}{2}}$, we obtain

$$\frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} = \sum_{\gamma\in H(\Gamma)}\sum_{n=1}^\infty \frac{\ell(\gamma)e^{-n\,\ell(\gamma)(v+k+1)}}{1-e^{-n\,\ell(\gamma)}} = \sum_{\gamma\in H(\Gamma)}\sum_{n=1}^\infty \frac{\ell(\gamma)e^{-n\,\ell(\gamma)\,\left(v+k+\frac{1}{2}\right)}}{2\sinh\left(\frac{n\,\ell(\gamma)}{2}\right)}. \quad (2.7.5)$$

Thus, we have

$$\frac{1}{s\Gamma(s)}\int_0^\infty \operatorname{HTr} K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t = \frac{1}{s\Gamma(s)\Gamma(1-s)}\int_0^\infty (v(v+2k+1))^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)}\mathrm{d}v.$$

By theorem 7.2 of [35], the function $\mathcal{Z}_\Gamma(s)$ has a simple zero at $s=1$ and it is non-zero at $s\in\mathbb{N}_{\geq 2}$. Let $a_{k,\Gamma}$ be the coefficient of the first term of the Laurent expansion of $\mathcal{Z}_\Gamma(v+k+1)$ at $v=0$, then

$$\mathcal{Z}_\Gamma(v+k+1) = a_{k,\Gamma}\, v^{N_k} + O_{k,\Gamma}(v^{N_k+1}) \qquad (v\to 0)\,,$$

where we recall that $N_0=1$ and $N_k=0$ for $k\geq 1$. Then, we can find functions $b_{k,\Gamma}(v)$, bounded in a neighborhood of $v=0$, such that

$$\frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} = \frac{N_k a_{k,\Gamma} v^{N_k-1} + O_{k,\Gamma}(v^{N_k})}{a_{k,\Gamma} v^{N_k} + O_{k,\Gamma}(v^{N_k+1})} = \frac{N_k}{v} + b_{k,\Gamma}(v). \tag{2.7.6}$$

Therefore, for $\eta\in\mathbb{R}_{>0}$ small, we can write the decomposition

$$\frac{1}{s\Gamma(s)}\int_0^\infty \operatorname{HTr} K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t + \frac{N_k}{s}$$

$$= \frac{1}{s\Gamma(s)\Gamma(1-s)}\int_0^\eta (v(v+2k+1))^{-s}\left(\frac{N_k}{v} + b_{k,\Gamma}(v)\right)\mathrm{d}v + \frac{N_k}{s} \tag{2.7.7}$$

$$+ \frac{1}{s\Gamma(s)\Gamma(1-s)}\int_\eta^\infty (v(v+2k+1))^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)}\mathrm{d}v, \tag{2.7.8}$$

where we have included the factor $\frac{N_k}{s}$ to have the complete the expression whose limit for $s\to 0^-$ we want to compute. We consider the limits for $s\to 0^-$ of the two terms separately, and we will then take the limit for $\eta\to 0$ of their sum. For the first term (2.7.7), using the boundedness of $b_{k,\Gamma}(v)$ in a neighborhood of $v=0$, we have

$$\lim_{s\to 0^-}\frac{1}{s\Gamma(s)\Gamma(1-s)}\int_0^\eta (v(v+2k+1))^{-s} b_{k,\Gamma}(v)\mathrm{d}v = \eta\, O_{k,\Gamma}(1),$$

because both the integral and the factor $\frac{1}{s\Gamma(s)\Gamma(1-s)}$ are holomorphic at $s=0$. We now consider the limit of the remaining part of (2.7.7), which is

$$\frac{1}{s\Gamma(s)\Gamma(1-s)}\int_0^\eta (v(v+2k+1))^{-s}\frac{N_k}{v}\mathrm{d}v + \frac{N_k}{s}. \tag{2.7.9}$$

Since $N_k=0$ for $k\neq 0$, we can set $k=0$ in the integral of the last expression. We then integrate it by parts

$$
\begin{aligned}
\int_0^\eta v^{-s-1}(v+1)^{-s}\mathrm{d}v &= \left(\frac{(v(v+1))^{-s}}{s}\right)_{v=0} - \left(\frac{(v(v+1))^{-s}}{s}\right)_{v=\eta} - \int_0^\eta v^{-s}(v+1)^{-s-1}\mathrm{d}v \\
&= -\frac{(\eta(\eta+1))^{-s}}{s} - \int_0^\eta v^{-s}(v+1)^{-s-1}\mathrm{d}v.
\end{aligned}
$$

We now consider the Taylor expansion at $s = 0$ of the last expression, observing that the integral is holomorphic at $s = 0$ and its domain of integration shrinks linearly in η. We find

$$
\begin{aligned}
\int_0^\eta v^{-s-1}(v+1)^{-s}\mathrm{d}v &= -\frac{e^{-s\log(\eta(\eta+1))}}{s} + \eta\, O(1) \\
&= -\frac{1}{s} + \log(\eta) + \log(\eta+1) + O_\eta(s) + \eta\, O(1) \qquad (s \to 0)\,.
\end{aligned}
$$

Replacing this expression in equation (2.7.9), we compute its limit for $s \to 0^-$

$$
\begin{aligned}
&\lim_{s\to 0^-} \frac{1}{s\Gamma(s)\Gamma(1-s)} \int_0^\eta (v(v+2k+1))^{-s}\frac{N_k}{v}\mathrm{d}v + \frac{N_k}{s} \\
&\qquad = \lim_{s\to 0^-} \frac{N_k}{s\Gamma(s)\Gamma(1-s)}\left(-\frac{1}{s} + \log(\eta) + \log(\eta+1) + \eta\, O(1) + O_\eta(s)\right) + \frac{N_k}{s} \\
&\qquad = N_k \log(\eta) + N_k\, \eta\, O(1),
\end{aligned}
$$

where we used the approximation $\log(\eta+1) \approx \eta$ for η small, and the limit

$$
\lim_{s\to 0^-} \frac{1}{s\Gamma(s)\Gamma(1-s)} = 1.
$$

Summing up, we have computed the limit of the term (2.7.7) to be

$$
\lim_{s\to 0^-} \frac{1}{s\Gamma(s)\Gamma(1-s)} \int_0^\eta (v(v+2k+1))^{-s}\left(\frac{N_k}{v} + b_{k,\Gamma}(v)\right)\mathrm{d}v + \frac{N_k}{s} = N_k\log(\eta) + \eta\, O_{k,\Gamma}(1).
$$

We now consider the limit of the second term (2.7.8) of the decomposition. First we observe that, since by equation (2.4.4) the lengths of closed geodesics $\ell(\gamma)$ are all positively uniformly bounded from below, the explicit expression (2.7.5) shows that the logarithmic derivative of the Selberg $\mathcal{Z}$-function has exponential decay for v large. In particular, the term (2.7.8) is holomorphic at $s = 0$. Using its absolute convergence, and the relations $\lim_{s\to 0} \frac{1}{s\Gamma(s)} = 1$ and $\Gamma(1) = 1$, we compute

$$\lim_{s\to 0^-} \frac{1}{s\Gamma(s)\Gamma(1-s)} \int_\eta^\infty (v(v+2k+1))^{-s} \frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} \mathrm{d}v$$

$$= \int_\eta^\infty \lim_{s\to 0^-} (v(v+2k+1))^{-s} \frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} \mathrm{d}v$$

$$= \int_\eta^\infty \frac{\frac{\mathrm{d}}{\mathrm{d}v}\mathcal{Z}_\Gamma(v+k+1)}{\mathcal{Z}_\Gamma(v+k+1)} \mathrm{d}v = -\log \mathcal{Z}_\Gamma(\eta+k+1).$$

Summing up, we have shown

$$\lim_{s\to 0^-} \frac{1}{s\Gamma(s)} \int_0^\infty \mathrm{HTr}\, K_k^\Gamma(t)\, t^{s-1}\mathrm{d}t + \frac{N_k}{s} = N_k \log(\eta) + \eta\, O_{k,\Gamma}(1) - \log \mathcal{Z}_\Gamma(\eta+k+1)$$

$$= \begin{cases} -\log\left(\frac{\mathcal{Z}_\Gamma(\eta+1)}{\eta}\right) + o(1), & k=0, \\ -\log \mathcal{Z}_\Gamma(\eta+k+1) + o(1), & k \geq 1, \end{cases} \qquad (\eta \to 0)\,.$$

Since $\mathcal{Z}_\Gamma(1) = 0$, taking the limit for $\eta\to 0$ proves the proposition. □

2.8 Computation of the identity contribution to the regularized determinant

The only term in the expression of the desired regularized determinant that remains implicit is the constant c_k. In this section, adapting a computation of Sarnak [55] to $k > 0$, we provide a formula for it.

Remark 2.8.1. The value of c_k appeared in several articles of the mathematical physics literature at the end of the 80s. In [50] Oshima observed that the value stated by D'Hoker and Phong [17, 18] is incorrect, but it can be deduced by comparing their work to a different computation of regularized determinants of Bolte and Steiner [8].

Before proving the proposition we need a preliminary lemma. Let us introduce the differential operator

$$\mathcal{D}_\nu(f)(\nu) := \left(\frac{\mathrm{d}}{\mathrm{d}\,(\nu(\nu+1))}\right)^2 f(\nu) = \frac{1}{2\nu+1}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{2\nu+1}\frac{\mathrm{d}}{\mathrm{d}\nu} f(\nu)\right).$$

Lemma 2.8.2. *Let $\nu \in \mathbb{R}$ and $g(t)$ be a continuous function on $\mathbb{R}_{>0}$ with asymptotic expansions*

$$g(t) = \begin{cases} \frac{a}{t} + b + O(t), & t\to 0, \\ O(e^{t\,d}), & t\to\infty, \end{cases} \qquad (a, b \in \mathbb{R};\ d \in \mathbb{R}_{>0})\,.$$

Then, we have the asymptotic expansion

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)_{s=0} = 2a\nu(\nu+1)\log(\nu) - a\nu^2 - 2b\log(\nu) + \frac{a}{2} + o(1) \qquad (\nu\to\infty)\,.$$

Moreover, we have the integral representation

$$\mathcal{D}_\nu\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)_{s=0}\right) = \int\limits_0^\infty e^{-t\nu(\nu+1)}g(t)\,t\,\mathrm{d}t \qquad \left(\nu > \frac{\sqrt{4d+1}-1}{2}\right).$$

Proof. We prove the former assertion first. We assume, without loss of generality, that $\mathrm{Re}(s) > -1$. Using the property of the Mellin transform

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{M}\left(f(t),s\right) = \mathcal{M}\left(\log(t)\,f(t),s\right), \tag{2.8.1}$$

we expand the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}&\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)\\ &= -\frac{\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)}{\Gamma(s)^2}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right) + \frac{1}{\Gamma(s)}\mathcal{M}\left(\log(t)e^{-t\nu(\nu+1)}g(t),s\right).\end{aligned}$$

Therefore, using the limit

$$\lim_{s\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)}{\Gamma(s)^2} = -1, \tag{2.8.2}$$

and adding and subtracting the asymptotic expansion of $g(t)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)_{s=0} = \lim_{s\to 0}\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right),s\right) \tag{2.8.3}$$

$$+\lim_{s\to 0}\left(\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right),s\right)\right) \tag{2.8.4}$$

$$+\lim_{s\to 0}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\log(t)e^{-t\nu(\nu+1)}g(t),s\right)\right). \tag{2.8.5}$$

We analyze the terms of the last sum separately. The first term (2.8.3) is holomorphic at $s=0$ by construction, thus

$$\lim_{s\to 0}\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right),s\right) = \int\limits_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\frac{\mathrm{d}t}{t}.$$

For the second term (2.8.4), we compute with $u = t\nu(\nu+1)$, to be

$$
\begin{aligned}
\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right),s\right) &= \int_0^\infty e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right)t^{s-1}\mathrm{d}t \\
&= a(\nu(\nu+1))^{1-s}\int_0^\infty e^{-u}u^{s-2}\mathrm{d}u + b(\nu(\nu+1))^{-s}\int_0^\infty e^{-u}u^{s-1}\mathrm{d}u \\
&= a(\nu(\nu+1))^{1-s}\Gamma(s-1) + b(\nu(\nu+1))^{-s}\Gamma(s), \qquad (2.8.6)
\end{aligned}
$$

where, even though the integral representations in between are only valid for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s) > 0$, respectively, the equality extends by analytic continuation to $\mathrm{Re}(s) > -1$. We further manipulate the third term (2.8.5) of the last decomposition by adding and subtracting the asymptotic expansion of $g(t)$ and the quantity

$$
\frac{\log(\nu(\nu+1))}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right),s\right),
$$

obtaining

$$
\begin{aligned}
\frac{1}{\Gamma(s)}\mathcal{M}\left(\log(t)e^{-t\nu(\nu+1)}g(t),s\right) = &\frac{1}{\Gamma(s)}\int_0^\infty \log(t)e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)t^{s-1}\mathrm{d}t \\
&+\frac{1}{\Gamma(s)}\mathcal{M}\left(\log(t\nu(\nu+1))e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right),s\right) \\
&-\frac{\log(\nu(\nu+1))}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}\left(\frac{a}{t}+b\right),s\right).
\end{aligned}
$$

The integral in the first term above is holomorphic, therefore the whole term goes to zero for $s \to 0$ because of the factor $\frac{1}{\Gamma(s)}$. The remaining two terms are analytically continued as in equation (2.8.6). Taking into account formula (2.8.1), this yields the following expression for the term (2.8.3)

$$
\begin{aligned}
&\frac{1}{\Gamma(s)}\mathcal{M}\left(\log(t)e^{-t\nu(\nu+1)}g(t),s\right) \\
&\qquad = o(1) + \frac{1}{\Gamma(s)}\left(a(\nu(\nu+1))^{1-s}\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s-1) + b(\nu(\nu+1))^{-s}\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)\right) \\
&\qquad\quad - \frac{\log(\nu(\nu+1))}{\Gamma(s)}\left(a(\nu(\nu+1))^{1-s}\Gamma(s-1) + b(\nu(\nu+1))^{-s}\Gamma(s)\right) \qquad (s \to 0).
\end{aligned}
$$

Therefore, composing the expressions for the terms (2.8.3), (2.8.4) and (2.8.5), and using the relation $\Gamma(s-1) = \frac{\Gamma(s)}{s-1}$, we find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)_{s=0}\\
&\qquad=\int_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\frac{\mathrm{d}t}{t}+\lim_{s\to 0}\Bigg(a(\nu(\nu+1))^{1-s}\frac{\Gamma(s)}{s-1}+b(\nu(\nu+1))^{-s}\Gamma(s)\\
&\qquad\qquad+\frac{1}{\Gamma(s)}\left(a(\nu(\nu+1))^{1-s}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Gamma(s)}{s-1}\right)+b(\nu(\nu+1))^{-s}\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)\right)\\
&\qquad\qquad-\frac{\log(\nu(\nu+1))}{\Gamma(s)}\left(a(\nu(\nu+1))^{1-s}\frac{\Gamma(s)}{s-1}+b(\nu(\nu+1))^{-s}\Gamma(s)\right)\Bigg).
\end{aligned}
$$

Since the Taylor series expansion at $s=0$ of the Γ-function is

$$\Gamma(s)=\frac{1}{s}-\gamma+O(s),$$

where γ denotes the Euler–Mascheroni constant, we have the limits

$$\lim_{s\to 0}\left(\Gamma(s)+\frac{\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)}{\Gamma(s)}\right)=0,$$

and

$$\lim_{s\to 0}\left(\frac{\Gamma(s)}{s-1}+\frac{\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Gamma(s)}{s-1}\right)}{\Gamma(s)}\right)=-1.$$

Using these limits and the asymptotic expansion

$$(\nu(\nu+1))^{-s}=e^{-s\log(\nu(\nu+1))}=1+O(s),$$

we take the limit for $s\to 0$ of the complicated expression above and conclude

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\nu(\nu+1)}g(t),s\right)\right)_{s=0}\\
&\qquad=\int_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\frac{\mathrm{d}t}{t}-a\nu(\nu+1)+(a\nu(\nu+1)-b)\log(\nu(\nu+1)).
\end{aligned}
$$

Applying the asymptotic expansion

$$\log(\nu(\nu+1))=2\log(\nu)+\log\left(1+\frac{1}{\nu}\right)=2\log(\nu)+\frac{1}{\nu}-\frac{1}{2\nu^2}+O\left(\frac{1}{\nu^3}\right)\qquad(\nu\to\infty)$$

to the last expression, we deduce the first claim of the lemma. To prove the second claim we apply the operator $\mathcal{D}_\nu$ to the right hand side of the last equation, observing that the assumption $\nu > \frac{\sqrt{4d+1}-1}{2}$ implies that the integrand is absolutely convergent and we can differentiate under the integral sign. For the integral term we obtain

$$\begin{aligned}
\mathcal{D}_\nu\left(\int_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\frac{\mathrm{d}t}{t}\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}\left(\nu(\nu+1)\right)}\right)^2 \int\limits_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\frac{\mathrm{d}t}{t} \\
&= \frac{\mathrm{d}}{\mathrm{d}\left(\nu(\nu+1)\right)}\left(-\int_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)\mathrm{d}t\right) \\
&= \int\limits_0^\infty e^{-t\nu(\nu+1)}\left(g(t)-\frac{a}{t}-b\right)t\,\mathrm{d}t.
\end{aligned}$$

For the remaining terms we have

$$\begin{aligned}
&\mathcal{D}_\nu\left(-a\nu(\nu+1)+\left(a\nu(\nu+1)-b\right)\log(\nu(\nu+1))\right) \\
&\qquad= \left(\frac{\mathrm{d}}{\mathrm{d}\left(\nu(\nu+1)\right)}\right)^2\left(-a\nu(\nu+1)+\left(a\nu(\nu+1)-b\right)\log(\nu(\nu+1))\right) \\
&\qquad= \frac{\mathrm{d}}{\mathrm{d}\left(\nu(\nu+1)\right)}\left(a\log(\nu(\nu+1))-\frac{b}{\nu(\nu+1)}\right) = \frac{a}{\nu(\nu+1)}+\frac{b}{(\nu(\nu+1))^2} \\
&\qquad= a\int\limits_0^\infty e^{-t\nu(\nu+1)}\mathrm{d}t + b\int\limits_0^\infty e^{-t\nu(\nu+1)}t\,\mathrm{d}t.
\end{aligned}$$

Summing up the two contributions proves the second claim of the lemma. □

Proposition 2.8.3. *The constants c_k have values*

$$c_k = \frac{\log(G(2k+1))}{2\pi} - \frac{2k-1}{4\pi}\log(\Gamma(2k+1)) + \frac{(2k+1)^2}{8\pi} - \frac{(2k+1)\log(2\pi)}{4\pi} - \frac{\zeta'(-1)}{\pi},$$

where $G(Z)$ denotes the Barnes G-function.

Proof. In analogy with the proof of proposition 2.7.2 the idea is to read $c_k \operatorname{vol}_{\mathrm{hyp}}(X)$ as the derivative of the identity contribution to the regularized determinant. Specifically, since $\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right)$ has a simple pole at $s=0$ with residue

$$-\frac{\operatorname{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi},$$

and, by definition, we have

$$c_k = \frac{1}{\operatorname{vol}_{\mathrm{hyp}}(X)}\left(\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right) + \frac{\operatorname{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi s}\right)_{s=0},$$

we have the Laurent expansion

$$\mathcal{M}\left(\operatorname{ITr} K_k^\Gamma(t), s\right) = -\frac{\operatorname{vol}_{\mathrm{hyp}}(X)\,(3k+1)}{12\pi s} + \operatorname{vol}_{\mathrm{hyp}}(X)\,c_k + O_{\Gamma,k}(s) \qquad (s\to 0)\,.$$

Therefore, we verify that

$$c_k = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(K_k(t;0),s\right)\right)_{s=0}. \tag{2.8.7}$$

We apply a deformation argument. Let us define the auxiliary function

$$\widetilde{c}_k(\nu) := \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t(\nu(\nu+1)-k(k+1))}K_k(t;0),s\right)\right)_{s=0} \qquad \left(\nu > \sqrt{k(k+1)}-\frac{1}{2}\right),$$

and observe that the choice $\nu = k$ satisfies the requirement of the formula and yields $\widetilde{c}_k(k) = c_k$.
The fact that $\widetilde{c}_k(\nu)$ is well-defined for $\nu > \sqrt{k(k+1)}-\frac{1}{2}$ follows from an application of the direct mapping theorem B.2 for the Mellin transform that uses the asymptotic expansions

$$e^{-t(\nu(\nu+1)-k(k+1))}K_k(t;0) = \begin{cases} \frac{1}{4\pi t}+\frac{1}{4\pi}\left(k^2-\frac{1}{3}-\nu(\nu+1)\right)+O(t), & t\to 0,\\ O\left(e^{-\left(\frac{1}{4}+\nu(\nu+1)-k(k+1)\right)t}\right), & t\to\infty, \end{cases} \tag{2.8.8}$$

obtained from formulae (2.6.1) and (2.6.3), respectively. From formula (2.6.1) we deduce the asymptotic expansion of $e^{tk(k+1)}K_k(t;0)$ for small t, namely

$$e^{tk(k+1)}K_k(t;0) = \frac{1}{4\pi t}+\frac{k^2-\frac{1}{3}}{4\pi}+O_k(t) \qquad (t\to 0).$$

Therefore we can apply the first statement of lemma 2.8.2 with parameters $a=\frac{1}{4\pi}$ and $b=\frac{k^2-\frac{1}{3}}{4\pi}$ to obtain the expansion

$$\begin{aligned}\widetilde{c}_k(\nu) &= \frac{1}{2\pi}\nu(\nu+1)\log(\nu)-\frac{\nu^2}{4\pi}-\frac{\left(k^2-\frac{1}{3}\right)}{2\pi}\log(\nu)+\frac{1}{8\pi}+o(1)\\ &= \frac{\nu^2\log(\nu)}{2\pi}-\frac{\nu^2}{4\pi}+\frac{\nu\log(\nu)}{2\pi}-\frac{\left(k^2-\frac{1}{3}\right)}{2\pi}\log(\nu)+\frac{1}{8\pi}+o(1) \qquad (\nu\to\infty).\end{aligned} \tag{2.8.9}$$

We now explicitly compute $\mathcal{D}_\nu(\widetilde{c}_k(\nu))$. By the integral representation of lemma 2.8.2 we have the relation

$$\mathcal{D}_\nu\left(\widetilde{c}_k(\nu)\right) = \int_0^\infty e^{-t(\nu(\nu+1)-k(k+1))}K_k(t;0)\,t\,\mathrm{d}t.$$

Applying formula (2.2.5), we deduce the explicit expression

$$\mathcal{D}_\nu\left(\widetilde{c}_k(\nu)\right) = \frac{1}{8\pi^{\frac{3}{2}}}\int_0^\infty\int_0^\infty \frac{e^{-t\left(\nu+\frac{1}{2}\right)^2-\frac{u^2}{4t}}}{\sqrt{t}}\frac{u\cosh(ku)}{\sinh\left(\frac{u}{2}\right)}\,\mathrm{d}u\,\mathrm{d}t.$$

By Fubini's theorem we exchange the order of integration. Then we perform the t-integration using formulae (2.7.2) and (2.7.3) as in the derivation of equality (2.7.4), we find

$$\int_0^\infty \frac{e^{-t\left(\nu+\frac{1}{2}\right)^2-\frac{u^2}{4t}}}{\sqrt{t}}\mathrm{d}t = \sqrt{\frac{4u}{2\nu+1}}K_{-\frac{1}{2}}\left(u\left(\nu+\frac{1}{2}\right)\right) = \frac{2\sqrt{\pi}}{2\nu+1}e^{-u\left(\nu+\frac{1}{2}\right)}.$$

Therefore, we have

$$\mathcal{D}_\nu\left(\widetilde{c}_k(\nu)\right)=\frac{1}{4\pi(2\nu+1)}\int\limits_0^\infty ue^{-u(\nu+1)}\frac{e^{ku}+e^{-ku}}{1-e^{-u}}\,\mathrm{d}u.$$

Formula 25.11.25 of [48] gives the following integral representation of the Hurwitz ζ-function

$$\zeta\left(Z,W\right)=\frac{1}{\Gamma(Z)}\int\limits_0^\infty\frac{u^{Z-1}e^{-Wu}}{1-e^{-u}}\mathrm{d}u\qquad\left(\mathrm{Re}(Z)>1,\,\mathrm{Re}(W)>1\right).$$

Since $\nu>\sqrt{k(k+1)}-\frac{1}{2}>k-1$, we can apply twice the formula above with $Z=2$ and $W=\nu\pm k+1$ to obtain

$$\mathcal{D}_\nu\left(\widetilde{c}_k(\nu)\right)=\frac{1}{4\pi(2\nu+1)}\left(\zeta(2,\nu+k+1)+\zeta(2,\nu-k+1)\right).\tag{2.8.10}$$

We now construct an explicit function, with the property that its image through $\mathcal{D}_\nu$ equals the right hand side of the formula above. This will establish an expression for $\widetilde{c}_k(\nu)$ up to polynomial terms in ν. For any $a\in\mathbb{R}_{>-\nu}$, we compute

$$\begin{aligned}&\mathcal{D}_\nu\left(\log\left(\frac{(2\pi)^\nu\Gamma(\nu+a)^{2a-3}}{G(\nu+a)^2}\right)-\nu(2a-2)\right)\\&\quad=\frac{1}{2\nu+1}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(\frac{1}{2\nu+1}\left(\log(2\pi)+(2a-3)\frac{\frac{\mathrm{d}}{\mathrm{d}\nu}\Gamma(\nu+a)}{\Gamma(\nu+a)}-\frac{2\frac{\mathrm{d}}{\mathrm{d}\nu}G(\nu+a)}{G(\nu+a)}-2a+2\right)\right).\end{aligned}$$

Differentiating formula 6.441.4 of [30], which states

$$\int\limits_0^Z\log(\Gamma(u+1))\mathrm{d}u=\frac{Z}{2}\log(2\pi)-\frac{Z(Z+1)}{2}+Z\log(\Gamma(Z+1))-\log(G(Z+1)),$$

we obtain the derivation formula for the Barnes G-function

$$\frac{\frac{\mathrm{d}}{\mathrm{d}Z}G(Z+1)}{G(Z+1)}=Z\frac{\frac{\mathrm{d}}{\mathrm{d}Z}\Gamma(Z+1)}{\Gamma(Z+1)}-Z+\frac{1}{2}\log(2\pi)-\frac{1}{2}.$$

We apply it to the preceding expression to obtain

$$\mathcal{D}_\nu\left(\log\left(\frac{(2\pi)^\nu\Gamma(\nu+a)^{2a-3}}{G(\nu+a)^2}\right)-\nu(2a-2)\right)=\frac{1}{2\nu+1}\frac{\mathrm{d}}{\mathrm{d}\nu}\left(-\frac{\frac{\mathrm{d}}{\mathrm{d}\nu}\Gamma(\nu+a)}{\Gamma(\nu+a)}+1\right).$$

The logarithmic derivative of the Γ-function is the so-called Digamma function ψ_0. By formula 5.15.1 of [48] its derivative is, by definition, a Hurwitz ζ-function. In formulae, we have

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left(-\frac{\frac{\mathrm{d}}{\mathrm{d}\nu}\Gamma(\nu+a)}{\Gamma(\nu+a)}\right)=-\frac{\mathrm{d}}{\mathrm{d}\nu}\psi_0(\nu+a)=-\sum_{n=0}^\infty\frac{1}{(n+\nu+a)^2}=-\zeta(2,\nu+a).$$

Combining the last two relations, with $a = 1 \pm k > -\nu$, and equation (2.8.10), we find

$$\begin{aligned}
-\frac{1}{4\pi}\mathcal{D}_\nu &\left(\log\left(\frac{(2\pi)^\nu\Gamma(\nu+k+1)^{2k-1}}{G(\nu+k+1)^2}\right) + \log\left(\frac{(2\pi)^\nu\Gamma(\nu-k+1)^{-2k-1}}{G(\nu-k+1)^2}\right)\right) \\
&= -\frac{1}{4\pi}\mathcal{D}_\nu\left(\log\left(\frac{(2\pi)^\nu\Gamma(\nu+k+1)^{2k-1}}{G(\nu+k+1)^2}\right) - 2\nu k\right) \\
&\quad - \frac{1}{4\pi}\mathcal{D}_\nu\left(\log\left(\frac{(2\pi)^\nu\Gamma(\nu-k+1)^{-2k-1}}{G(\nu-k+1)^2}\right) + 2\nu k\right) \\
&= \frac{1}{4\pi(2\nu+1)}\left(\zeta(2,\nu+k+1) + \zeta(2,\nu-k+1)\right) = \mathcal{D}_\nu(\widetilde{c}_k(\nu)).
\end{aligned}$$

Inverting the differential operator $\mathcal{D}_\nu$, this relation implies

$$\begin{aligned}
\widetilde{c}_k(\nu) = &-\frac{1}{4\pi}\log\left(\frac{(2\pi)^\nu\Gamma(\nu+k+1)^{2k-1}}{G(\nu+k+1)^2}\right) - \frac{1}{4\pi}\log\left(\frac{(2\pi)^\nu\Gamma(\nu-k+1)^{-2k-1}}{G(\nu-k+1)^2}\right) \\
&+ \nu(\nu+1)K_1 + K_2,
\end{aligned} \tag{2.8.11}$$

where $K_1, K_2 \in \mathbb{R}$ are unknown constants. The last step in the computation is to read off the constants by comparing the asymptotic expansion of $\nu \to \infty$ of the two sides. We quote the asymptotic expansion of the logarithm of the Γ-function, which reads

$$\log\left(\Gamma(Z+1)\right) = \left(Z + \frac{1}{2}\right)\log(Z+1) - Z - 1 + \frac{\log(2\pi)}{2} + O\left(\frac{1}{Z}\right)$$
$$(Z \in \mathbb{C},\ |\arg(Z)| < \pi,\ |Z| \to \infty).$$

Moreover, formula 5.17.5 of [48] gives the asymptotic expansion of the logarithm of the Barnes G-function, namely

$$\log\left(G(Z+1)\right) = \frac{1}{2}Z^2\log(Z) - \frac{3}{4}Z^2 + \frac{\log(2\pi)}{2}Z - \frac{1}{12}\log(Z) + \zeta'(-1) + O\left(\frac{1}{Z}\right),$$
$$(|\arg(Z)| < \pi,\ |Z| \to \infty),$$

where we have written $\zeta'(-1)$ in place of $\frac{1}{12} - \log(A)$, with A the Glaisher–Kinkelin constant, according to formula 5.17.7 of loc. cit.. Finally, we observe the expansion of the logarithm

$$\log(\nu+b) = \log(\nu) + \log\left(1 + \frac{b}{\nu}\right) = \log(\nu) + \frac{b}{\nu} - \frac{b^2}{2\nu^2} + O_b\left(\frac{1}{\nu^3}\right) \qquad (b \in \mathbb{R},\ \nu \to \infty)\,.$$

Applying these relations to formula (2.8.11) we obtain the asymptotic expansion

$$\begin{aligned}
\widetilde{c}_k(\nu) =& \frac{1}{2\pi}\log(G(\nu+k+1)) + \frac{1}{2\pi}\log(G(\nu-k+1)) - \nu\frac{\log(2\pi)}{2\pi} \\
&- \frac{2k-1}{4\pi}\log(\Gamma(\nu+k+1)) + \frac{2k+1}{4\pi}\log(\Gamma(\nu-k+1)) + \nu(\nu+1)K_1 + K_2
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2\pi}\left(\frac{(\nu+k)^2}{2}\log(\nu+k)-\frac{3}{4}(\nu+k)^2+\frac{\log(2\pi)}{2}(\nu+k)-\frac{\log(\nu+k)}{12}+\zeta'(-1)\right)\\
&+\frac{1}{2\pi}\left(\frac{(\nu-k)^2}{2}\log(\nu-k)-\frac{3}{4}(\nu-k)^2+\frac{\log(2\pi)}{2}(\nu-k)-\frac{\log(\nu-k)}{12}+\zeta'(-1)\right)\\
&-\nu\frac{\log(2\pi)}{2\pi}-\frac{2k-1}{4\pi}\left(\left(\nu+k+\frac{1}{2}\right)\log(\nu+k+1)-\nu-k-1+\frac{\log(2\pi)}{2}\right)\\
&+\frac{2k+1}{4\pi}\left(\left(\nu-k+\frac{1}{2}\right)\log(\nu-k+1)-\nu+k-1+\frac{\log(2\pi)}{2}\right)+\nu(\nu+1)K_1\\
&+K_2+o(1)\\
=&\frac{1}{2\pi}\left(\nu^2\log(\nu)-\frac{3}{2}\nu^2+\nu\log(2\pi)+\left(k^2-\frac{1}{6}\right)\log(\nu)-\frac{k^2}{2}+2\zeta'(-1)\right)\\
&-\nu\frac{\log(2\pi)}{2\pi}-\frac{2k-1}{4\pi}\left(\nu\log(\nu)-\nu+\left(k+\frac{1}{2}\right)\log(\nu)+\frac{\log(2\pi)}{2}\right)\\
&+\frac{2k+1}{4\pi}\left(\nu\log(\nu)-\nu-\left(k-\frac{1}{2}\right)\log(\nu)+\frac{\log(2\pi)}{2}\right)+\nu(\nu+1)K_1+K_2+o(1)\\
=&\frac{\nu^2\log(\nu)}{2\pi}+\left(K_1-\frac{3}{4\pi}\right)\nu^2+\frac{\nu\log(\nu)}{2\pi}+\left(K_1-\frac{1}{2\pi}\right)\nu+\frac{1-3k^2}{6\pi}\log(\nu)+K_2\\
&+\frac{\log(2\pi)}{4\pi}+\frac{\zeta'(-1)}{\pi}+o(1)\qquad(\nu\to\infty)\,.
\end{aligned}
$$

Comparing the last expression with the expansion given in formula (2.8.9) we deduce the values

$$
K_1=\frac{1}{2\pi},\qquad K_2=-\frac{\log(2\pi)}{4\pi}-\frac{\zeta'(-1)}{\pi}+\frac{1}{8\pi},
$$

therefore we have

$$
\begin{aligned}
\widetilde{c}_k(\nu)=&\frac{1}{2\pi}\log(G(\nu+1+k))+\frac{1}{2\pi}\log(G(\nu+1-k))-\nu\frac{\log(2\pi)}{2\pi}\\
&-\frac{2k-1}{4\pi}\log(\Gamma(\nu+1+k))+\frac{2k+1}{4\pi}\log(\Gamma(\nu+1-k))+\frac{\nu(\nu+1)}{2\pi}-\frac{\log(2\pi)}{4\pi}\\
&-\frac{\zeta'(-1)}{\pi}+\frac{1}{8\pi}.
\end{aligned}
$$

By construction we have the equality $c_k=\widetilde{c}_k(k)$. Since $G(1)=\Gamma(1)=1$, we find

$$
\begin{aligned}
c_k=&\frac{1}{2\pi}\log(G(2k+1))-k\frac{\log(2\pi)}{2\pi}-\frac{2k-1}{4\pi}\log(\Gamma(2k+1))+\frac{k(k+1)}{2\pi}-\frac{\log(2\pi)}{4\pi}\\
&-\frac{\zeta'(-1)}{\pi}+\frac{1}{8\pi},
\end{aligned}
$$

rearranging terms the proof of the proposition is complete. □

For the sake of completeness we collect the main results of the chapter in a final theorem.

Theorem 2.8.4. *The regularized determinant of the Laplacian $\Delta^1_{S_{k+1}}$ is given by the expression*

$$\det{}^*_\Gamma\left(\Delta^1_{S_{k+1}}\right) = \begin{cases} \mathcal{Z}'_\Gamma(1)\, e^{-c_0 \operatorname{vol}_{\mathrm{hyp}}(X)}\, 2^{\frac{\operatorname{vol}_{\mathrm{hyp}}(X)}{6\pi}+2}, & k=0, \\ \mathcal{Z}_\Gamma(k+1)\, e^{-c_k \operatorname{vol}_{\mathrm{hyp}}(X)}\, 2^{\frac{(3k+1)\operatorname{vol}_{\mathrm{hyp}}(X)}{6\pi}}, & k \geq 1. \end{cases}$$

Here $\mathcal{Z}_\Gamma(s)$ is the Selberg zeta function associated to Γ, and

$$c_k = \frac{\log(G(2k+1))}{2\pi} - \frac{2k-1}{4\pi}\log(\Gamma(2k+1)) + \frac{(2k+1)^2}{8\pi} - \frac{(2k+1)\log(2\pi)}{4\pi} - \frac{\zeta'(-1)}{\pi},$$

where $G(Z)$ denotes the Barnes G-function.

Proof. In view of definitions 2.6.1 and 2.6.5, the statement is a combination of observation 2.6.6, and propositions 2.7.2 and 2.8.3. □

Chapter 3

The heat kernel on the model cusp

In this chapter we present an explicit formula for the heat kernel of weight k on the model cusp $\Gamma_\infty\backslash\mathbb{H}$. The expression is analogous to a formula for the heat kernel of weight 0 for a cusp on an arbitrary closed manifold provided by Müller [46].

3.1 Definition of the heat kernel on the model cusp

In the notation introduced in section (2.5), we have

$$\begin{aligned}K_k^{\infty}(t;z,z) &= \sum_{n\in\mathbb{Z}} \left(\frac{z+n-\bar{z}}{z-\bar{z}-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n)) \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n)). \qquad (3.1.1)\end{aligned}$$

This function is the object of study of this chapter. Using formula (2.1.1) we introduce the variable

$$u := \cosh(d_{\mathrm{hyp}}(z,z+n)) = 1+\frac{n^2}{2y^2}.$$

Moreover, we observe the relations

$$\frac{u-1}{u+1} = \frac{\left(\frac{n}{2y}\right)^2}{\left(\frac{n}{2y}\right)^2+1}, \qquad \frac{2}{u+1} = \frac{1}{\left(\frac{n}{2y}\right)^2+1}, \qquad \left(\frac{2iy+n}{2iy-n}\right)^k = (-1)^k\left(\frac{\frac{n}{2y}+i}{\frac{n}{2y}-i}\right)^k.$$

Now, formulae (2.2.6) and (2.2.7) imply

$$\begin{aligned}&K_k(t;d_{\mathrm{hyp}}(z,z+n)) \\ &= \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t(2k-j)(j+1)} \left(\frac{2}{1+u}\right)^{k-j} {}_2F_1\left(-j,2k-j;1;\frac{u-1}{u+1}\right) \\ &\quad+ \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi}\int_0^\infty r\tanh(\pi r)e^{-tr^2}\left(\frac{2}{1+u}\right)^{\frac{1}{2}+ir} {}_2F_1\left(-k+\frac{1}{2}+ir,k+\frac{1}{2}+ir;1;\frac{u-1}{u+1}\right)\mathrm{d}r.\end{aligned}$$

Therefore, for $w \in \mathbb{R}$ we write

$$h_k(t,w) := (-1)^k \left(\frac{w+i}{w-i}\right)^k \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t(2k-j)(j+1)} \left(\frac{1}{w^2+1}\right)^{k-j} \times {}_2F_1\left(-j, 2k-j; 1; \frac{w^2}{w^2+1}\right), \tag{3.1.2}$$

and

$$f_k(t,w) := \frac{(-1)^k e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \left(\frac{w+i}{w-i}\right)^k \int_0^\infty r \tanh(\pi r) e^{-tr^2} \left(\frac{1}{w^2+1}\right)^{\frac{1}{2}+ir} \times {}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1; \frac{w^2}{w^2+1}\right) \mathrm{d}r, \tag{3.1.3}$$

and we find the expression

$$K_k^\infty(t;z,z) = \sum_{n\in\mathbb{Z}} \left(h_k\left(t, \frac{n}{2y}\right) + f_k\left(t, \frac{n}{2y}\right)\right). \tag{3.1.4}$$

Lemma 3.1.1. *Let $t \in \mathbb{R}_{>0}$ be fixed, then, as functions of $w \in \mathbb{R}$, we have*

$$h_k(t,w),\ f_k(t,w) \in L^1(\mathbb{R}).$$

Proof. Directly from its definition we have

$$\|h_k(t,w)\|_{L^1} \leq \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t(2k-j)(j+1)} \int_{-\infty}^{\infty} \left(\frac{1}{w^2+1}\right)^{k-j} \left|{}_2F_1\left(-j, 2k-j; 1; \frac{w^2}{w^2+1}\right)\right| \mathrm{d}w.$$

By formula 15.2.4 of [48], since $j \in \mathbb{N}$, the hypergeometric function under consideration is given by the finite sum

$${}_2F_1\left(-j, 2k-j; 1; \frac{w^2}{w^2+1}\right) = \sum_{m=0}^{j} \frac{(-1)^m \Gamma(j+1)\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(2k-j)\Gamma(m+1)^2} \left(\frac{w^2}{w^2+1}\right)^m. \tag{3.1.5}$$

Therefore, we have the estimate

$$\|h_k(t,w)\|_{L^1} \leq \sum_{j=0}^{k-1} \frac{2k-2j-1}{4\pi} e^{-t(2k-j)(j+1)} \sum_{m=0}^{j} \frac{\Gamma(j+1)\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(2k-j)\Gamma(m+1)^2} \times \int_{-\infty}^{\infty} \left(\frac{1}{w^2+1}\right)^{k-j} \left(\frac{w^2}{w^2+1}\right)^m \mathrm{d}w.$$

Since each of the finite number of integrals occurring in the last expression is convergent, the statement for $h_k(t,w)$ follows. We now examine the coefficients $f_k(t,w)$. By construction we have the relation

$$h_k(t,w)+f_k(t,w)=(-1)^k\left(\frac{w+i}{w-i}\right)^k K_k(t;\operatorname{arccosh}(1+2w^2)),$$

that implies the bound

$$\|f_k(t,w)\|_{L^1}\le\|h_k(t,w)\|_{L^1}+\int\limits_{-\infty}^{\infty}K_k(t;\operatorname{arccosh}(1+2w^2))\mathrm{d}w.$$

We already showed that $\|h_k(t,w)\|_{L^1}$ is finite. Since $K_k(t;\operatorname{arccosh}(1+2w^2))$ is even and continuous, we only have to verify the convergence of the integral above for w large and positive. Assume $w>1$, then $\operatorname{arccosh}(1+w^2)>1$ and we can apply the bound on $K_k(t;d)$ given in proposition 2.3.4 with $\delta=1$. We find

$$K_k(t;\operatorname{arccosh}(1+2w^2))\ll_{k,t}e^{-\frac{(\operatorname{arccosh}(1+2w^2))^2}{4t}}.$$

Since $\operatorname{arccosh}(1+2w^2)\ge\log(w)$ for $w\ge1$, we conclude

$$K_k(t;\operatorname{arccosh}(1+2w^2))\ll_{k,t}w^{-\frac{\log(w)}{4t}}.$$

This proves the claimed integrability because the exponent of the last expression goes to $-\infty$ for $w\to\infty$ and t fixed. $\square$

By lemma 3.1.1 the Fourier transforms

$$\widehat{h}_k(t,v):=\int\limits_{-\infty}^{\infty}h_k(t,w)e^{-iwv}\mathrm{d}w\qquad(v\in\mathbb{R}),$$

and

$$\widehat{f}_k(t,v):=\int\limits_{-\infty}^{\infty}f_k(t,w)e^{-iwv}\mathrm{d}w\qquad(v\in\mathbb{R}),$$

are well-defined.
Formula (3.1.4) and the Poisson summation formula, theorem 4.2.8 in [51], imply the relation

$$K_k^{\infty}(t;z,z)=2y\sum_{n\in\mathbb{Z}}\left(\widehat{h}_k(t,4\pi yn)+\widehat{f}_k(t,4\pi yn)\right),\tag{3.1.6}$$

which is for the moment only formally true, since the absolute convergence of its right hand side has not been established. We refer to the $\widehat{h}_k(t,v)$ as the *discrete coefficients* and to the $\widehat{f}_k(t,v)$ as the *continuous coefficients* of the summation formula for the heat kernel.

3.2 Computation of the discrete coefficients

In this section we provide explicit formulae for the coefficients $\widehat{h}_k(t,v)$, distinguishing the case $v=0$ from the case $v\neq 0$.

Lemma 3.2.1. *The coefficients $\widehat{h}_k(t,v)$ are identically zero for $v>0$, and, for $v<0$, they have the explicit expression*

$$\widehat{h}_k(t,v) = -\frac{1}{2v}\sum_{j=0}^{k-1}\frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\Gamma(j+1)}W_{k,k-j-\frac{1}{2}}(-v)^2, \tag{3.2.1}$$

where $W_{\kappa,\mu}(Z)$ is the Whittaker W-function.

Proof. The binomial expansion of $(w^2+1-1)^m$ implies the identity

$$\frac{w^{2m}}{(w^2+1)^m} = \sum_{l=0}^{m}(-1)^l\binom{m}{l}\left(\frac{1}{w^2+1}\right)^l = \sum_{l=0}^{m}(-1)^l\frac{\Gamma(m+1)}{\Gamma(m-l+1)\Gamma(l+1)}\left(\frac{1}{w^2+1}\right)^l.$$

Combining the definition of $\widehat{h}_k(t,v)$ with formula (3.1.5), and then applying the identity above, we obtain

$$\begin{aligned}\widehat{h}_k(t,v) =&(-1)^k\sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t(2k-j)(j+1)}\sum_{m=0}^{j}\frac{(-1)^m\Gamma(j+1)\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(2k-j)\Gamma(m+1)^2}\\ &\times\int_{-\infty}^{\infty}\left(\frac{w+i}{w-i}\right)^k\left(\frac{1}{w^2+1}\right)^{k-j}\left(\frac{w^2}{w^2+1}\right)^m e^{-iwv}\,\mathrm{d}w \quad (3.2.2)\\ =&(-1)^k\sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t(2k-j)(j+1)}\frac{\Gamma(j+1)}{\Gamma(2k-j)}\sum_{m=0}^{j}\frac{(-1)^m\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(m+1)}\\ &\times\sum_{l=0}^{m}\frac{(-1)^l}{\Gamma(m-l+1)\Gamma(l+1)}\int_{-\infty}^{\infty}\left(\frac{w+i}{w-i}\right)^k\left(\frac{1}{w^2+1}\right)^{k-j+l}e^{-iwv}\mathrm{d}w.\end{aligned}$$

The integral can be explicitly computed via formula 3.384.9 of [30], namely

$$\begin{aligned}&\int_{-\infty}^{\infty}(\beta+iw)^{-2\mu}(\gamma-iw)^{-2\nu}e^{-iwv}\mathrm{d}w \quad (3.2.3)\\ &=\begin{cases}2\pi(\beta+\gamma)^{-\mu-\nu}\frac{v^{\mu+\nu-1}}{\Gamma(2\nu)}e^{\frac{\beta-\gamma}{2}v}W_{\nu-\mu,\frac{1}{2}-\nu-\mu}(\beta v+\gamma v), & v>0,\\ 2\pi(\beta+\gamma)^{-\mu-\nu}\frac{(-v)^{\mu+\nu-1}}{\Gamma(2\mu)}e^{\frac{\beta-\gamma}{2}v}W_{\mu-\nu,\frac{1}{2}-\nu-\mu}(-\beta v-\gamma v), & v<0,\end{cases}\\ &\left(\mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0,\ \mathrm{Re}(\mu+\nu)>\frac{1}{2}\right).\end{aligned}$$

Therefore, if $v > 0$, we have

$$\int_{-\infty}^{\infty} \left(\frac{w+i}{w-i}\right)^k \left(\frac{1}{w^2+1}\right)^{k-j+l} e^{-iwv}\mathrm{d}w = (-1)^k \int_{-\infty}^{\infty} (1+iw)^{-2k+j-l}(1-iw)^{j-l}e^{-iwv}\mathrm{d}w$$
$$= (-1)^k\, \pi \frac{\left(\frac{v}{2}\right)^{k-j+l-1}}{\Gamma(-j+l)} W_{-k,-k+j-l+\frac{1}{2}}(2v) = 0,$$

because $\frac{1}{\Gamma(-j+l)}$ vanishes for $l \in \{0, ..., j\}$. While, if $v < 0$, we obtain

$$\int_{-\infty}^{\infty} \left(\frac{w+i}{w-i}\right)^k \left(\frac{1}{w^2+1}\right)^{k-j+l} e^{-iwv}\mathrm{d}w = (-1)^k \int_{-\infty}^{\infty} (1+iw)^{-2k+j-l}(1-iw)^{j-l}e^{-iwv}\mathrm{d}w$$
$$= (-1)^k\, \pi \frac{\left(-\frac{v}{2}\right)^{k-j+l-1}}{\Gamma(2k-j+l)} W_{k,-k+j-l+\frac{1}{2}}(-2v).$$

Thus, assuming $v < 0$ and using the connection formula 13.14.31 of [48], which states

$$W_{\kappa,\mu}(Z) = W_{\kappa,-\mu}(Z) \qquad (\kappa, \mu, Z \in \mathbb{C}),$$

we have

$$\widehat{h}_k(t,v) = \sum_{j=0}^{k-1} \frac{2k-2j-1}{4} e^{-t(2k-j)(j+1)} \frac{\Gamma(j+1)}{\Gamma(2k-j)} \sum_{m=0}^{j} \frac{(-1)^m\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(m+1)}$$
$$\times \sum_{l=0}^{m} \frac{(-1)^l \left(-\frac{v}{2}\right)^{k-j+l-1}}{\Gamma(m-l+1)\Gamma(l+1)\Gamma(2k-j+l)} W_{k,k-j+l-\frac{1}{2}}(-2v).$$

The statement of equation (3.2.1) is now equivalent to the summation formula

$$\sum_{m=0}^{j} \frac{(-1)^m\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(m+1)} \sum_{l=0}^{m} \frac{(-1)^l \left(-\frac{v}{2}\right)^{k-j+l}}{\Gamma(m-l+1)\Gamma(l+1)\Gamma(2k-j+l)} W_{k,k-j+l-\frac{1}{2}}(-2v)$$
$$= \left(\frac{W_{k,k-j-\frac{1}{2}}(-v)}{\Gamma(j+1)}\right)^2. \tag{3.2.4}$$

The Whittaker W-function is related to the generalized Laguerre polynomials by formula 13.18.19 of [48], namely

$$W_{\frac{\alpha+1}{2}+n,\frac{\alpha}{2}}(Z) = (-1)^n\Gamma(n+1)Z^{\frac{\alpha+1}{2}} L_n^{(\alpha)}(Z) \qquad (n \in \mathbb{N};\ \alpha, Z \in \mathbb{C}). \tag{3.2.5}$$

Applying it we reduce to prove

$$\sum_{m=0}^{j} \frac{(-1)^{m+j}\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(m+1)} \sum_{l=0}^{m} \frac{\Gamma(j-l+1)(-v)^{2l}}{\Gamma(m-l+1)\Gamma(l+1)\Gamma(2k-j+l)} L_{j-l}^{(2k-2j+2l-1)}(-2v)$$
$$= L_j^{(2k-2j-1)}(-v)^2.$$

We rewrite the left hand side of the last expression using $a = j - m$ and $b = j - l$. After rearranging terms the equality to be proven becomes

$$\sum_{b=0}^{j} \frac{\Gamma(b+1)(-v)^{2j-2b}}{\Gamma(j-b+1)\Gamma(2k-b)} L_b^{(2k-2b-1)}(-2v) \sum_{a=0}^{b} \frac{(-1)^a \Gamma(2k-a)}{\Gamma(a+1)\Gamma(j-a+1)\Gamma(b-a+1)} = L_j^{(2k-2j-1)}(-v)^2.$$

We now explicitly compute the a-summation. Let us recall that the Pochhammer symbol $(a)_n$ is defined by the assignment

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (n \in \mathbb{N};\ a \in \mathbb{C}).$$

Moreover, we use the definition of the hypergeometric function, formula 15.2.4 of [48], and the Chu–Vandermonde identity, formula 15.4.24 of loc. cit., to write

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{(r)_l}{(s)_l} = {}_2F_1(-n, r; s; 1) = \frac{(s-r)_n}{(s)_n} \qquad (n \in \mathbb{N};\ r, s \in \mathbb{C}), \tag{3.2.6}$$

and we finally recall the binomial identity

$$\binom{r}{s} = (-1)^s \binom{s-r-1}{s} \qquad (r, s \in \mathbb{C}),$$

which is obtained by expanding the binomials in terms of Γ-functions and using their functional equation. Let $c := b - a$, we compute

$$\begin{aligned}
\sum_{a=0}^{b} \frac{(-1)^a \Gamma(2k-a)}{\Gamma(a+1)\Gamma(j-a+1)\Gamma(b-a+1)} &= (-1)^b \sum_{c=0}^{b} \frac{(-1)^c \Gamma(2k-b+c)}{\Gamma(b-c+1)\Gamma(j-b+c+1)\Gamma(c+1)} \\
&= \frac{(-1)^b \Gamma(2k-b)}{\Gamma(b+1)\Gamma(j-b+1)} \sum_{c=0}^{b} (-1)^c \binom{b}{c} \frac{(2k-b)_c}{(j-b+1)_c} \\
&= \frac{(-1)^b \Gamma(2k-b)}{\Gamma(b+1)\Gamma(j-b+1)} {}_2F_1(-b, 2k-b; j-b+1; 1) \\
&= \frac{(-1)^b \Gamma(2k-b)}{\Gamma(b+1)\Gamma(j-b+1)} \frac{(-2k+j+1)_b}{(j-b+1)_b} \\
&= \frac{(-1)^b \Gamma(2k-b)}{\Gamma(j+1)} \frac{\Gamma(-2k+j+b+1)}{\Gamma(b+1)\Gamma(-2k+j+1)} \\
&= \frac{\Gamma(2k-b)}{\Gamma(j+1)} (-1)^b \binom{-2k+j+b}{b} \\
&= \frac{\Gamma(2k-b)\Gamma(2k-j)}{\Gamma(j+1)\Gamma(2k-j-b)\Gamma(b+1)}.
\end{aligned}$$

We are thus reduced to prove

$$\sum_{b=0}^{j} \frac{(-v)^{2j-2b}\, \Gamma(2k-j)}{\Gamma(j-b+1)\, \Gamma(j+1)\Gamma(2k-j-b)} L_b^{(2k-2b-1)}(-2v) = L_j^{(2k-2j-1)}(-v)^2,$$

which follows directly from formula 8.976.4 of [30], namely

$$L_n^{(\alpha)}(Z)L_n^{(\alpha)}(W)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\sum_{l=0}^{n}\frac{L_{n-l}^{(\alpha+2l)(Z+W)}}{\Gamma(l+\alpha+1)}\frac{(ZW)^l}{\Gamma(l+1)}.$$

This proves the summation formula (3.2.4) and equation (3.2.1). □

Lemma 3.2.2. *The discrete coefficients have, for $v=0$, the values*

$$\widehat{h}_k(t,0)=0.$$

Proof. Setting $v=0$ in formula (3.2.2) we find

$$\widehat{h}_k(t,0)=(-1)^k\sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t(2k-j)(j+1)}\sum_{m=0}^{j}\frac{(-1)^m\Gamma(j+1)\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(2k-j)\Gamma(m+1)^2}$$
$$\times\int_{-\infty}^{\infty}\left(\frac{w+i}{w-i}\right)^k\left(\frac{1}{w^2+1}\right)^{k-j}\left(\frac{w^2}{w^2+1}\right)^m \mathrm{d}w.$$

We write

$$\left(\frac{w+i}{w-i}\right)^k=\frac{(w+i)^{2k}}{(w^2+1)^k}$$

and observe that

$$\mathrm{Re}\left(\frac{(w+i)^{2k}}{(w^2+1)^k}\right)=\sum_{l=0}^{k}\binom{2k}{2l}\frac{w^{2l}(-1)^{k-l}}{(w^2+1)^k},\tag{3.2.7}$$

and

$$\mathrm{Im}\left(\frac{(w+i)^{2k}}{(w^2+1)^k}\right)=\sum_{l=0}^{k-1}\binom{2k}{2l+1}\frac{w^{2l+1}(-1)^{k-l}}{(w^2+1)^k},\tag{3.2.8}$$

are an even and an odd function, respectively. Therefore, since the remaining part of the integrand is even, we ignore the contribution of the imaginary part. Further expanding the binomial in terms of Γ-functions leads to

$$\widehat{h}_k(t,0)=(-1)^k\sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t(2k-j)(j+1)}\sum_{m=0}^{j}\frac{(-1)^m\Gamma(j+1)\Gamma(2k-j+m)}{\Gamma(j-m+1)\Gamma(2k-j)\Gamma(m+1)^2}$$
$$\times\sum_{l=0}^{k}\frac{(-1)^{k-l}\Gamma(2k+1)}{\Gamma(2k-2l+1)\Gamma(2l+1)}\int_{-\infty}^{\infty}\frac{w^{2l}}{(w^2+1)^k}\frac{1}{(w^2+1)^{k-j}}\left(\frac{w^2}{w^2+1}\right)^m\mathrm{d}w.$$

In the w-integral occurring in the last expression we apply the change of variable $v=\frac{w^2}{w^2+1}$, observing $1-v=\frac{1}{w^2+1}$ and $\mathrm{d}w=\frac{(w^2+1)^2\mathrm{d}v}{2w}$, to obtain

$$\int_{-\infty}^{\infty}\frac{w^{2(m+l)}}{(w^2+1)^{2k-j+m}}\mathrm{d}w=\frac{1}{2}\int_0^1 v^{m+l-\frac{1}{2}}(1-v)^{2k-j-l-\frac{3}{2}}\mathrm{d}v.$$

Now, using formula 3.191.3 of [30], namely

$$\int_0^1 v^{\mu-1}(1-v)^{\nu-1}\mathrm{d}v = B(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \qquad (\mathrm{Re}(\mu)>0,\, \mathrm{Re}(\nu)>0)\,,$$

we explicitly compute the last integral. We find

$$\int_{-\infty}^{\infty} \frac{w^{2(m+l)}}{(w^2+1)^{2k-j+m}}\,\mathrm{d}w = \frac{1}{2}\frac{\Gamma\left(m+l+\frac{1}{2}\right)\Gamma\left(2k-j-l-\frac{1}{2}\right)}{\Gamma(2k+m-j)}.$$

Thus, simplifying and rearranging terms, we find

$$\begin{aligned}\widehat{h}_k(t,0) =& \sum_{j=0}^{k-1}\frac{2k-2j-1}{8\pi}e^{-t(2k-j)(j+1)}\sum_{l=0}^{k}(-1)^l\frac{\Gamma(j+1)\Gamma(2k+1)\Gamma\left(2k-j-l-\frac{1}{2}\right)}{\Gamma(2k-j)\Gamma(2k-2l+1)\Gamma(2l+1)}\\ &\times\sum_{m=0}^{j}(-1)^m\frac{\Gamma\left(m+l+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)^2}.\end{aligned}$$

The value of the latter sum can be given explicitly. To do so, we first write it as a hypergeometric function and then we apply the Chu–Vandermonde identity, equation (3.2.6). We compute

$$\begin{aligned}\sum_{m=0}^{j}(-1)^m\frac{\Gamma\left(m+l+\frac{1}{2}\right)}{\Gamma(j-m+1)\Gamma(m+1)^2} &= \frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(j+1)}\sum_{m=0}^{j}(-1)^m\binom{j}{m}\frac{\left(l+\frac{1}{2}\right)_m}{(1)_m}\\ &= \frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(j+1)}\,{}_2F_1\left(-j,l+\frac{1}{2};1;1\right)\\ &= \frac{\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(j+1)}\frac{\left(-l+\frac{1}{2}\right)_j}{(1)_j}\\ &= \frac{\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(j-l+\frac{1}{2}\right)}{\Gamma(j+1)^2\Gamma\left(-l+\frac{1}{2}\right)}.\end{aligned}\tag{3.2.9}$$

Rearranging terms, we deduce

$$\begin{aligned}\widehat{h}_{t,k}(0) =& \sum_{j=0}^{k-1}\frac{2k-2j-1}{4\pi}e^{-t(2k-j)(j+1)}\frac{\Gamma(2k+1)}{\Gamma(2k-j)\Gamma(j+1)}\\ &\times\sum_{l=0}^{k}(-1)^l\frac{\Gamma\left(2k-j-l-\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(j-l+\frac{1}{2}\right)}{\Gamma(2k-2l+1)\Gamma(2l+1)\Gamma\left(-l+\frac{1}{2}\right)}.\end{aligned}$$

To complete the computation it is now enough to show that the latter sum is zero for any $j\in\{0,...,k-1\}$. We define

$$\Sigma_k(j) := \sum_{l=0}^{k}(-1)^l\frac{\Gamma\left(2k-j-l-\frac{1}{2}\right)\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(j-l+\frac{1}{2}\right)}{\Gamma(2k-2l+1)\Gamma(2l+1)\Gamma\left(-l+\frac{1}{2}\right)}.$$

We substitute $m = k - l$, to obtain

$$\Sigma_k(j) = (-1)^k \sum_{m=0}^{k} (-1)^m \frac{\Gamma\left(k+m-j-\frac{1}{2}\right)\Gamma\left(k-m+\frac{1}{2}\right)\Gamma\left(-k+j+m+\frac{1}{2}\right)}{\Gamma(2m+1)\Gamma(2k-2m+1)\Gamma\left(-k+m+\frac{1}{2}\right)},$$

then we use the duplication formula for the Γ-function, formula 5.5.5 of [48], which reads

$$\Gamma(2Z) = \frac{2^{2Z-1}}{\sqrt{\pi}}\Gamma(Z)\Gamma\left(Z+\frac{1}{2}\right) \qquad (2Z \notin \mathbb{Z}_{\leq 0}), \tag{3.2.10}$$

on the terms of the denominator containing $2m$. We find

$$\Sigma_k(j) = \left(-\frac{1}{4}\right)^k \pi \sum_{m=0}^{k} (-1)^m \frac{\Gamma\left(k+m-j-\frac{1}{2}\right)\Gamma\left(-k+j+m+\frac{1}{2}\right)}{\Gamma(m+1)\Gamma\left(m+\frac{1}{2}\right)\Gamma(k-m+1)\Gamma\left(-k+m+\frac{1}{2}\right)}.$$

Rewriting this expression in terms of Pochhammer symbols we see that it is of the form of a generalized hypergeometric function, namely

$$\begin{aligned}&\Sigma_k(j)\\ &= \left(-\frac{1}{4}\right)^k \pi \frac{\Gamma\left(k-j-\frac{1}{2}\right)\Gamma\left(-k+j+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-k+\frac{1}{2}\right)} {}_3F_2\left(-k, k-j-\frac{1}{2}, -k+j+\frac{1}{2}; \frac{1}{2}, \frac{1}{2}-k; 1\right).\end{aligned}$$

The function ${}_3F_2$ is of the form prescribed by Saalschütz's theorem, which is given as formula 16.4.3 of [48] and states

$${}_3F_2(-n, a, b; c, 1+a+b-c-n; 1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \qquad (n \in \mathbb{N};\ a, b, c \in \mathbb{C}). \tag{3.2.11}$$

Applying it we find

$$\Sigma_k(j) = \left(-\frac{1}{4}\right)^k \pi \frac{\Gamma\left(k-j-\frac{1}{2}\right)\Gamma\left(-k+j+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-k+\frac{1}{2}\right)} \frac{(-k+j+1)_k\,(k-j)_k}{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}.$$

Expanding the Pochammer symbols in terms of Γ-functions, we conclude

$$\Sigma_k(j) = \left(-\frac{1}{4}\right)^k \pi \frac{\Gamma\left(k-j-\frac{1}{2}\right)\Gamma\left(-k+j+\frac{1}{2}\right)\Gamma(j+1)\Gamma(2k-j)\Gamma\left(\frac{1}{2}\right)^2}{\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)\Gamma\left(-k+\frac{1}{2}\right)\Gamma(-k+j+1)\Gamma(k-j)\Gamma\left(k+\frac{1}{2}\right)^2}.$$

All the Γ-functions in the last expression are evaluated at points where the Γ-function is holomorphic, except at $-k+j+1$ where, for any $j \in \{0, ..., k-1\}$, it has a pole. Therefore

$$\Sigma_k(j) = 0,$$

and the proof of the lemma is complete. □

3.3 Computation of the continuous coefficients

In this section we explicitly compute the continuous coefficients $\widehat{f_k}(t,v)$ for $v \in \mathbb{R}$. As we did for the discrete coefficients, we distinguish the cases $v = 0$ and $v \neq 0$.

Lemma 3.3.1. *Let $v \in \mathbb{R}_{>0}$, then*

$$\widehat{f_k}(t,\pm v) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi^2 v} \int_0^\infty r \sinh(2\pi r) e^{-tr^2} \left| \Gamma\left(\pm k + \frac{1}{2} + ir\right)\right|^2 W_{\mp k, ir}(v)^2 \, \mathrm{d}r. \tag{3.3.1}$$

Proof. For $\eta \in \left(0, \frac{1}{4}\right)$ we define the perturbed term

$$\widehat{f_k}(t,\pm v,\eta) := \int_{-\infty}^{\infty} \left(\frac{1}{w^2+1}\right)^{\eta} f_k(t,w)\, e^{\mp iwv} \mathrm{d}w.$$

Using $\|f_k(t,w)\|_{L^1} < \infty$ to apply Lebesgue's theorem, we find

$$\widehat{f_k}(t,\pm v) = \lim_{\eta \to 0^+} \widehat{f_k}(t,\pm v,\eta).$$

We now compute the term

$$\begin{aligned}\widehat{f_k}(t,\pm v,\eta) = & \frac{(-1)^k e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \int_{-\infty}^{\infty} \left(\frac{w+i}{w-i}\right)^k \int_0^\infty r \tanh(\pi r) e^{-tr^2} \left(\frac{1}{w^2+1}\right)^{\eta+\frac{1}{2}+ir} \\ & \times {}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1; \frac{w^2}{w^2+1}\right) e^{\mp iwv} \, \mathrm{d}r \mathrm{d}w.\end{aligned}$$

We exchange the order of integration. To justify it via Fubini's theorem we have to prove the convergence of

$$\int_{-\infty}^{\infty} \int_0^\infty \left| r \tanh(\pi r) e^{-tr^2} \left(\frac{1}{w^2+1}\right)^{\eta} P_{\frac{1}{2}+ir,k}(2w^2+1) \right| \mathrm{d}r \, \mathrm{d}w.$$

The only term of the integrand which is not real and positive is the second modified Legendre function. In the third line of [23, pg 151][1] Fay states the following integral representation

$$P_{s,k}(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1-e^{i\theta}\rho}{1-e^{-i\theta}\rho}\right)^k \left(\frac{1-\rho^2}{\left|e^{i\theta}-\rho\right|^2}\right)^s \mathrm{d}\theta,$$

[1]We remark that Fay [23] writes as $P_{s,k}(d)$ what we, and Oshima [49], write as $P_{s,k}(\cosh(d))$. We point out that our notation is consistent with the notation for Legendre polynomials, and it is already in use in the statement of the next formula.

where $\rho = \tanh\left(\frac{\operatorname{arccosh}(r)}{2}\right)$. Combining it with the relation

$$\tanh\left(\frac{\operatorname{arccosh}(2w^2+1)}{2}\right)^2 = \frac{w^2}{w^2+1},$$

obtained from formula 4.35.22 of [48], we have an explicit formula for the absolute value we want to estimate. Namely

$$\left|P_{\frac{1}{2}+ir,k}(2w^2+1)\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-e^{i\theta}\sqrt{\frac{w^2}{w^2+1}}}{1-e^{-i\theta}\sqrt{\frac{w^2}{w^2+1}}}\right)^k \left(\frac{1-\frac{w^2}{w^2+1}}{\left|e^{i\theta}-\frac{w^2}{w^2+1}\right|^2}\right)^{\frac{1}{2}+ir} \mathrm{d}\theta\right|.$$

Further moving the absolute value into the integral, we find

$$\left|P_{\frac{1}{2}+ir,k}(2w^2+1)\right| \ll \left(\frac{1}{w^2+1}\right)^{\frac{1}{2}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{\left|e^{i\theta}-\frac{w^2}{w^2+1}\right|}.$$

Using formulae 2.571.4 and 2.571.5 of [30], which state

$$\begin{aligned}
\int \frac{\mathrm{d}v}{\sqrt{a-b\cos(v)}} &= \frac{2}{\sqrt{a+b}} F\left(\arcsin\sqrt{\frac{(a+b)(1-\cos(v))}{2(a-b\cos(v))}}, \sqrt{\frac{2b}{a+b}}\right),\\
\int \frac{\mathrm{d}v}{\sqrt{a+b\cos(v)}} &= \frac{2}{\sqrt{a+b}} F\left(\frac{v}{2}, \sqrt{\frac{2b}{a+b}}\right) \qquad (0<b<a,\ 0\le v\le \pi),
\end{aligned}$$

the latter integral can be written as a combination of elliptic integrals. We find

$$\begin{aligned}
\int_0^{2\pi} \frac{\mathrm{d}\theta}{\left|e^{i\theta}-\frac{w^2}{w^2+1}\right|} &= \int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{1+\left(\frac{w^2}{w^2+1}\right)^2 - 2\frac{w^2}{w^2+1}\cos(\theta)}} + \int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{1+\left(\frac{w^2}{w^2+1}\right)^2 + 2\frac{w^2}{w^2+1}\cos(\theta)}}\\
&= \frac{4w^2+4}{2w^2+1}\left(F\left(\frac{\pi}{2}, \frac{2w\sqrt{w^2+1}}{2w^2+1}\right) - F\left(0, \frac{2w\sqrt{w^2+1}}{2w^2+1}\right)\right).
\end{aligned}$$

The second elliptic integral is 0 by definition. According to the definition 13.7 (1) of [19], the first one is complete and it can be estimated through formula 13.8 (9) of [19], namely

$$F\left(\frac{\pi}{2}, a\right) + \log\sqrt{1-a^2} \le \frac{\pi}{2} \qquad (a\in[0,1)).$$

This yields the inequality

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\left|e^{i\theta}-\frac{w^2}{w^2+1}\right|} \le \frac{4w^2+4}{2w^2+1}\left(\frac{\pi}{2} + \log\left(2w^2+1\right)\right).$$

Replacing this value into the expression we have to estimate and simplifying, we deduce

$$\left| P_{\frac{1}{2}+ir,k}(2w^2+1) \right| \ll \frac{1+\log(w^2+1)}{\sqrt{w^2+1}}. \tag{3.3.2}$$

Since $\eta > 0$, the convergence that had to be proven is now elementary. Exchanging the integrals we obtain

$$\begin{aligned} \widehat{f_k}(t, \pm v, \eta) = \frac{(-1)^k e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \int\limits_0^\infty r \tanh(\pi r) e^{-tr^2} \int\limits_{-\infty}^\infty \left(\frac{w+i}{w-i}\right)^k \qquad\qquad \\ \times \left(\frac{1}{w^2+1}\right)^{\eta+\frac{1}{2}+ir} {}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1; \frac{w^2}{w^2+1}\right) e^{\mp iwv} \, \mathrm{d}w \, \mathrm{d}r. \end{aligned} \tag{3.3.3}$$

A direct application of a transformation formula for hypergeometric functions, formula 15.3.6 of [1], implies

$$\begin{aligned} &{}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1; \frac{w^2}{w^2+1}\right) \\ &\quad = \frac{\Gamma(-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)} {}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1+2ir; \frac{1}{w^2+1}\right) \\ &\quad + \left(\frac{1}{w^2+1}\right)^{-2ir} \frac{\Gamma(2ir)}{\Gamma\left(-k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}+ir\right)} \\ &\qquad\qquad \times {}_2F_1\left(k+\frac{1}{2}-ir, -k+\frac{1}{2}-ir; 1-2ir; \frac{1}{w^2+1}\right). \end{aligned}$$

This, together with the notation,

$$\begin{aligned} \varphi_k(w, r, \eta) := \left(\frac{w+i}{w-i}\right)^k \left(\frac{1}{w^2+1}\right)^{\eta+\frac{1}{2}+ir} \frac{\Gamma(-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)} \\ \times {}_2F_1\left(-k+\frac{1}{2}+ir, k+\frac{1}{2}+ir; 1+2ir; \frac{1}{w^2+1}\right), \end{aligned}$$

leads to the formula

$$\widehat{f_k}(t, \pm v, \eta) = \frac{(-1)^k e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \int\limits_{-\infty}^\infty r \tanh(\pi r) e^{-tr^2} \int\limits_{-\infty}^\infty \varphi_k(w, r, \eta) e^{\mp iwv} \, \mathrm{d}w \, \mathrm{d}r. \tag{3.3.4}$$

We now compute the w-integral. Using the power series expansion of the hypergeometric function, formula 15.2.1 of [48], we find

$$\int_{-\infty}^{\infty} \varphi_k(w,r,\eta)e^{\mp iwv}\mathrm{d}w$$

$$= \frac{\Gamma(-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)} \int_{-\infty}^{\infty} (w+i)^{k-\eta-\frac{1}{2}-ir}(w-i)^{-k-\eta-\frac{1}{2}-ir}$$
$$\times \sum_{j=0}^{\infty} \frac{\left(-k+\frac{1}{2}+ir\right)_j \left(k+\frac{1}{2}+ir\right)_j}{(1+2ir)_j\,\Gamma(j+1)}(w-i)^{-j}(w+i)^{-j}e^{\mp iwv}\mathrm{d}w.$$

We exchange the order of series and integration. To justify this, expanding the Pochhammer symbols and excluding the value $j=0$, we have to prove the convergence of

$$\sum_{j=1}^{\infty}\int_{-\infty}^{\infty} \left|\frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)\,\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)\,\Gamma(j+1)}\right| \left(\frac{1}{w^2+1}\right)^{j+\eta+\frac{1}{2}} \mathrm{d}w.$$

For j fixed, the family of w-integrals is increasing for $\eta\to 0^+$, and for $\eta=0$ it can be explicitly integrated. To see this we now prove the differentiation formula

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\sum_{m=0}^{j-1}\left(\prod_{l=1}^{m}\frac{2(j-l)}{2l+1}\right)\frac{w^{2m+1}}{(w^2+1)^{j-\frac{1}{2}}}\right) = \left(\frac{1}{w^2+1}\right)^{j+\frac{1}{2}} \qquad (j\in\mathbb{Z}_{\geq 1})\,. \tag{3.3.5}$$

Expanding the derivative of the left hand side we find

$$\frac{\mathrm{d}}{\mathrm{d}w}\left(\sum_{m=0}^{j-1}\left(\prod_{l=1}^{m}\frac{2(j-l)}{2l+1}\right)\frac{w^{2m+1}}{(w^2+1)^{j-\frac{1}{2}}}\right)$$
$$= \left(\frac{1}{w^2+1}\right)^{j+\frac{1}{2}} \sum_{m=0}^{j-1}\left(\prod_{l=1}^{m}\frac{2(j-l)}{2l+1}\right)\left((-2j+2m+2)w^{2m+2}+(2m+1)w^{2m}\right).$$

In the last sum, for $m=j-1$ the coefficient of w^{2j} is 0, and for $m=0$ the coefficient of w^0 is 1. Therefore it is enough to show that for any $h\in\{0,...,j-2\}$ the coefficients of w^{2h+2} coming from $m=h$ and $m=h+1$ cancel each other. This is equivalent to

$$\left(\prod_{l=1}^{h}\frac{2(j-l)}{2l+1}\right)(-2j+2h+2) + \left(\prod_{l=1}^{h+1}\frac{2(j-l)}{2l+1}\right)(2h+3) = 0,$$

which is easily verified. Therefore formula (3.3.5) is established, and it implies

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}w}{(w^2+1)^{j+\frac{1}{2}}} = 2\prod_{l=1}^{j-1}\frac{2(j-l)}{2l+1} = 2^j(j-1)!\prod_{l=1}^{j-1}\frac{1}{2l+1}$$
$$= 2^{2j-1}\frac{(j-1)!\,(j-1)!}{(2j-1)!} = 2^{2j-1}\frac{\Gamma(j)^2}{\Gamma(2j)}.$$

Applying Stirling's formula, given as expression 5.11.3 of [48] and which states

$$\Gamma(Z) \sim \sqrt{2\pi} e^{-Z} Z^{Z-\frac{1}{2}} \left(1 + O\left(\frac{1}{Z}\right)\right) \qquad (|\arg(Z)| < \pi,\ |Z| \to \infty)\,,$$

proves the asymptotic expansion

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}w}{(w^2+1)^{j+\frac{1}{2}}} = \sqrt{\frac{\pi}{j}} + O\left(j^{-\frac{3}{2}}\right) \qquad (j \to \infty)\,.$$

On the other hand, formulae 5.6.6 and 5.6.7 of [48], namely

$$|\Gamma(\mathrm{Re}(Z))| \geq |\Gamma(Z)| \geq \sqrt{\mathrm{sech}(\pi \mathrm{Im}(Z))}\, \Gamma(\mathrm{Re}(Z)) \qquad \left(\mathrm{Re}(Z) \geq \frac{1}{2}\right), \tag{3.3.6}$$

and formula 6.1.47 of [1], i.e.,

$$\frac{\Gamma(v+a)}{\Gamma(v+b)} \sim v^{a-b} \qquad \left(v \in \mathbb{R}_{>\max\{|a|,\,|b|\}},\ v \to \infty\right), \tag{3.3.7}$$

imply

$$\begin{aligned} \left|\frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)\,\Gamma(j+1)}\right| &\leq \sqrt{\cosh(2\pi r)} \left|\frac{\Gamma\left(-k+j+\frac{1}{2}\right)\Gamma\left(k+j+\frac{1}{2}\right)}{\Gamma(j+1)^2}\right| \\ &\ll_k \frac{\sqrt{\cosh(2\pi r)}}{j}. \end{aligned}$$

This proves the necessary absolute convergence to exchange the w-integral and the j-series. Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi_k(w,r,\eta) e^{\mp iwv} \mathrm{d}w = {} & \frac{\Gamma(-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)} \sum_{j=0}^{\infty} \frac{\left(-k+\frac{1}{2}+ir\right)_j \left(k+\frac{1}{2}+ir\right)_j}{(1+2ir)_j\,\Gamma(j+1)} \\ & \times \int_{-\infty}^{\infty} (w+i)^{k-j-\eta-\frac{1}{2}-ir} (w-i)^{-k-j-\eta-\frac{1}{2}-ir} e^{\mp iwv} \mathrm{d}w. \end{aligned} \tag{3.3.8}$$

Recalling the assumption $\eta > 0$, the latter integral can be related, through equation (3.2.3), to a Whittaker function. Specifically, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} (w+i)^{k-j-\eta-\frac{1}{2}-ir} (w-i)^{-k-j-\eta-\frac{1}{2}-ir} e^{\mp iwv} \mathrm{d}w \\ & \qquad = (-1)^k \int_{-\infty}^{\infty} (1+iw)^{-k-j-\eta-\frac{1}{2}-ir} (1-iw)^{k-j-\eta-\frac{1}{2}-ir} e^{\mp iwv} \mathrm{d}w \\ & \qquad = (-1)^k \pi \left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}+ir} \frac{W_{\mp k,\, j+\eta+ir}(2v)}{\Gamma\left(\mp k+j+\eta+\frac{1}{2}+ir\right)}. \end{aligned}$$

This leads to the formula

$$
\begin{aligned}
&\widehat{f}_k(t,\pm v,\eta)\\
&= \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2}\int\limits_{-\infty}^{\infty} r\tanh(\pi r)e^{-tr^2}\frac{\Gamma(-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)}\\
&\qquad\times\sum_{j=0}^{\infty}\frac{\left(-k+\frac{1}{2}+ir\right)_j\left(k+\frac{1}{2}+ir\right)_j}{(1+2ir)_j\,\Gamma(j+1)}\left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}+ir}\frac{W_{\mp k,\,j+\eta+ir}(2v)}{\Gamma\left(\mp k+j+\eta+\frac{1}{2}+ir\right)}\,\mathrm{d}r\\
&= \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2}\int\limits_{-\infty}^{\infty} r\tanh(\pi r)\frac{e^{-tr^2}\,\Gamma(-2ir)\Gamma(1+2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}+ir\right)}\\
&\quad\times\sum_{j=0}^{\infty}\frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)\,\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)\,\Gamma(j+1)}\left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}+ir}\frac{W_{\mp k,\,j+\eta+ir}(2v)}{\Gamma\left(\mp k+j+\eta+\frac{1}{2}+ir\right)}\,\mathrm{d}r,
\end{aligned}
$$

where in the second equality we expanded the Pochhammer symbols. Formulae 6.1.29 and 6.1.30 of [1] state

$$
\Gamma(-ir)\Gamma(ir) = \frac{\pi}{r\sinh(\pi r)},\qquad \Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{1}{2}+ir\right) = \frac{\pi}{\cosh(\pi r)}\qquad (r\in\mathbb{R}). \tag{3.3.9}
$$

Combining them with the functional equation of the Γ-function we obtain the formulae

$$
\begin{aligned}
\Gamma(-2ir)\Gamma(1+2ir) &= \frac{i\pi}{\sinh(2\pi r)},\\
\Gamma\left(-k+\frac{1}{2}-ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}+ir\right) &= \frac{\pi^2}{\cosh(\pi r)^2}.
\end{aligned}
$$

Using these relations, together with the duplication formula for the sinh-function, yields

$$
\begin{aligned}
\widehat{f}_k(t,\pm v,\eta) = \frac{ie^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi}\int\limits_{-\infty}^{\infty} re^{-tr^2}\sum_{j=0}^{\infty}\frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)\Gamma\left(j+1\right)}&\\
\times\left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}+ir}\frac{W_{\mp k,\,j+\eta+ir}(2v)}{\Gamma\left(\mp k+j+\eta+\frac{1}{2}+ir\right)}\,\mathrm{d}r.&
\end{aligned}
$$

We claim that the limit $\eta\to 0^+$ can be carried in the r-integration and the j-summation. This is implied by the existence of a uniform η-bound on the absolute value of the last expression, which we now prove. First, we need bounds on the absolute value of the Whittaker function. Let $a,b,c,d,v\in\mathbb{R}$ be such that $a-c<\frac{1}{2}$ and $v>0$. By the integral representation 13.16.5 of [48], namely

$$
W_{\kappa,\mu}(Z) = \frac{Z^{\mu+\frac{1}{2}}2^{-2\mu}}{\Gamma\left(\mu-\kappa+\frac{1}{2}\right)}\int\limits_{1}^{\infty} e^{-\frac{uZ}{2}}(u-1)^{\mu-\kappa-\frac{1}{2}}(u+1)^{\mu+\kappa-\frac{1}{2}}\mathrm{d}u\qquad\left(\mathrm{Re}\,(\kappa-\mu)<\frac{1}{2}\right), \tag{3.3.10}
$$

and by the inequalities (3.3.6), we have

$$|W_{a+ib,\,c+id}(v)| \le \left| \frac{\Gamma\left(c-a+\frac{1}{2}\right)}{\Gamma\left(c-a+\frac{1}{2}+i(d-b)\right)} \right| W_{a,\,c}(v) \le \sqrt{\cosh(\pi(d-b))}\, W_{a,\,c}(v). \tag{3.3.11}$$

This bound covers all the cases that we will need later, except $W_{k,\,j+\eta+ir}(2v)$ when $j<k$. To handle these cases, which for k fixed are finitely many, we observe that a repeated application of the recurrence relation 13.15.11 of [48]

$$W_{\kappa+1,\mu}(Z) + (2\kappa - Z)W_{\kappa,\mu}(Z) + \left(\kappa-\mu-\frac{1}{2}\right)\left(\kappa+\mu-\frac{1}{2}\right) W_{\kappa-1,\mu}(Z) = 0, \tag{3.3.12}$$

shows that

$$W_{k,\,j+\eta+ir}(2v) = p_k(k, j+\eta+ir, v)\, W_{0,\,j+\eta+ir}(2v) + q_k(k, j+\eta+ir, v)\, W_{-1,\,j+\eta+ir}(2v), \tag{3.3.13}$$

where p_k and q_k are polynomials with rational coefficients of degrees bounded by $k+1$ in the variables k, $j+\eta+ir$ and v. We now prove the claimed bound for the case where the first parameter of the Whittaker W-function is $+k$, the other case is easier since the bound (3.3.11) applies to every term of the j-sum. The combination of the estimates

$$\left| \frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)}{\Gamma\left(k+j+\eta+\frac{1}{2}+ir\right)} \right| \le e^{\frac{\pi r}{2}} \left| \frac{\Gamma\left(-k+j+\frac{1}{2}\right)}{\Gamma\left(k+j+\eta+\frac{1}{2}\right)} \right|,$$

and

$$\left| \frac{\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)} \right| \le e^{\pi r} \frac{\Gamma\left(k+j+\frac{1}{2}\right)}{\Gamma(j+1)},$$

given by the inequalities (3.3.6), with formulae (3.3.11) and (3.3.13) implies

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \left| re^{-tr^2} \frac{\Gamma\left(-k+j+\frac{1}{2}+ir\right)\Gamma\left(k+j+\frac{1}{2}+ir\right)}{\Gamma(j+1+2ir)\Gamma\left(j+1\right)\Gamma\left(k+j+\eta+\frac{1}{2}+ir\right)} \left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}+ir} \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \times W_{k,\,j+\eta+ir}(2v) \right| \mathrm{d}r \\
&\quad \le \sum_{j=k}^{\infty} \left| \frac{\Gamma\left(-k+j+\frac{1}{2}\right)}{\Gamma(j+1)^2} \frac{\Gamma\left(k+j+\frac{1}{2}\right)}{\Gamma\left(k+j+\eta+\frac{1}{2}\right)} \right| \left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}} W_{k,\,j+\eta}(2v) \int_{-\infty}^{\infty} re^{-tr^2+2\pi r} \mathrm{d}r \\
&\qquad + \sum_{j=0}^{k-1} \left| \frac{\Gamma\left(-k+j+\frac{1}{2}\right)}{\Gamma(j+1)^2} \frac{\Gamma\left(k+j+\frac{1}{2}\right)}{\Gamma\left(k+j+\eta+\frac{1}{2}\right)} \right| \left(\frac{v}{2}\right)^{j+\eta-\frac{1}{2}} \left(W_{0,\,j+\eta}(2v) + W_{-1,\,j+\eta}(2v)\right) \\
&\qquad\qquad \times \int_{-\infty}^{\infty} \left(|p_k(k, j+\eta+ir, v)| + |q_k(k, j+\eta+ir, v)|\right) re^{-tr^2+2\pi r} \mathrm{d}r.
\end{aligned}$$

To start proving the uniform η-bounds on the infinite j-sum and on the finite j-sum, we observe the estimate

$$\left|\frac{\Gamma\left(\frac{1}{2}+k+j\right)}{\Gamma\left(\frac{1}{2}+k+j+\eta\right)}\right| \ll 1 \qquad (j \in \mathbb{N}),$$

which is obtained by monotonicity for $k \geq 1$ or $j \geq 1$, and by substituting $k = j = 0$ in the expression otherwise. Formula 13.20.2 of [48] states

$$W_{\kappa,\mu}(v) = \frac{\Gamma(\kappa+\mu)}{\sqrt{\pi}} \left(\frac{v}{4}\right)^{-\mu+\frac{1}{2}} \left(1+O\left(\mu^{-1}\right)\right) \qquad \left(|\arg(\mu)| < \frac{\pi}{2},\ |\mu| \to \infty\right)$$

uniformly for bounded positive real values v. Therefore, for v and k fixed, we have the asymptotic expansion

$$W_{k,\,j+\eta}(2v) = \frac{\Gamma(k+j+\eta)\left(\frac{v}{2}\right)^{-j-\eta+\frac{1}{2}}}{\sqrt{\pi}} \left(1+O_{k,v}(j^{-1})\right) \qquad (j \to \infty).$$

Since, by equation (3.3.7) we have

$$\frac{\Gamma\left(-k+j+\frac{1}{2}\right)\Gamma(k+j+\eta)}{\Gamma(j+1)^2} = j^{\eta-\frac{3}{2}} + O_k\left(j^{\eta-\frac{5}{2}}\right) \qquad (j \to \infty),$$

the infinite j-sum is uniformly convergent for $\eta \in \left[0, \frac{1}{4}\right)$. It remains to prove a η-uniform bound on the finite j-sum. Given that the degree of the polynomials p_k and q_k with respect to r is bounded by $k+1$, that j is bounded and that v is fixed, the r-integral in each term of the finite sum is convergent and uniformly bounded for j in the range of the sum. Therefore, since the Whittaker W-function is entire in the second parameter, the finite sum is uniformly bounded for any $\eta \in \left[0, \frac{1}{4}\right)$.
Performing the limit for $\eta \to 0^+$ into the r-integral and the j-summation we conclude

$$\widehat{f}_k(t, \pm v) = \frac{ie^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi} \int_{-\infty}^{\infty} re^{-tr^2} \sum_{j=0}^{\infty} \frac{\Gamma\left(\pm k+j+\frac{1}{2}+ir\right)\left(\frac{v}{2}\right)^{j-\frac{1}{2}+ir}}{\Gamma(j+1+2ir)\Gamma(j+1)} W_{\mp k,\,j+ir}(2v)\,\mathrm{d}r.$$

We now further simplify this expression. For any $\kappa \in \mathbb{C}$ and $\xi \in \mathbb{R}_{>0}$ we define

$$g_\xi(\kappa, t, v) := \left(\int_{-\infty}^{-\xi} + \int_{\xi}^{\infty}\right) re^{-tr^2} \sum_{j=0}^{\infty} \frac{\Gamma\left(\kappa+j+\frac{1}{2}+ir\right)\left(\frac{v}{2}\right)^{j-\frac{1}{2}+ir}}{\Gamma(j+1+2ir)\Gamma(j+1)} W_{-\kappa,\,j+ir}(2v)\,\mathrm{d}r.$$

Then we observe the relation

$$-4\pi i\, e^{t\left(k+\frac{1}{2}\right)^2} \widehat{f}_k(t, \pm v) = \lim_{\xi \to 0} g_\xi(\pm k, t, v).$$

We claim that $g_\xi(\kappa,t,v)$ is holomorphic for κ in the domain

$$\left\{\kappa\in\mathbb{C}\,\middle|\,\mathrm{Re}(\kappa)>-\frac{1}{2}\right\}\cup\left\{\kappa\in\mathbb{C}\,\middle|\,|\mathrm{Im}(\kappa)|<\xi\right\}.$$

Indeed, as shown above, $g_\xi(\kappa,t,v)$ is an absolutely convergent integral of an absolutely convergent sum of functions that are holomorphic for κ in the above mentioned domain, because the Whittaker function $W_{\kappa,\,j+ir}(v)$ is entire in κ for any j,r,v fixed, as stated after formula 13.14.13 in [48], and $\Gamma(Z)$ is holomorphic at Z if $\mathrm{Re}(Z)>0$ or if $\mathrm{Im}(Z)\neq 0$. For the upcoming manipulations we restrict to $\mathrm{Re}(\kappa)>-\frac{1}{2}$. Applying the integral representation of the Whittaker function given by equation (3.3.10), we find

$$\begin{aligned}&g_\xi(\kappa,t,v)\\&=\left(\int_{-\infty}^{-\xi}+\int_{\xi}^{\infty}\right)re^{-tr^2}\sum_{j=0}^{\infty}\frac{2\left(\frac{v}{2}\right)^{2j+2ir}}{\Gamma(j+1+2ir)\Gamma\left(j+1\right)}\int_1^\infty e^{-vu}(u^2-1)^{j-\frac{1}{2}+ir}\left(\frac{u-1}{u+1}\right)^{\kappa}\mathrm{d}u\,\mathrm{d}r.\end{aligned}$$

Applying the change of variables $w=\operatorname{arccosh}(u)$, with $\mathrm{d}w=\frac{\mathrm{d}u}{\sqrt{u^2-1}}$, and using the absolute convergence to exchange the j-summation and the w-integral, we obtain

$$\begin{aligned}&g_\xi(\kappa,t,v)\\&=2\left(\int_{-\infty}^{-\xi}+\int_{\xi}^{\infty}\right)re^{-tr^2}\int_0^\infty e^{-v\cosh(w)}\left(\frac{\cosh(w)-1}{\cosh(w)+1}\right)^{\kappa}\sum_{j=0}^{\infty}\frac{\left(\frac{v\sinh(w)}{2}\right)^{2j+2ir}}{\Gamma(j+1+2ir)\Gamma\left(j+1\right)}\mathrm{d}w\,\mathrm{d}r.\end{aligned}$$

Formula 10.25.2 of [48] gives a series representation for the modified Bessel function of the first kind, namely

$$I_\nu(Z)=\left(\frac{Z}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{\left(\frac{Z}{2}\right)^{2n}}{\Gamma(\nu+n+1)\Gamma(n+1)}\qquad(Z\in\mathbb{C}),$$

Thus we have

$$g_\xi(\kappa,t,v)=2\left(\int_{-\infty}^{-\xi}+\int_{\xi}^{\infty}\right)re^{-tr^2}\int_0^\infty e^{-v\cosh(w)}\left(\frac{\cosh(w)-1}{\cosh(w)+1}\right)^{\kappa}I_{2ir}(v\sinh(w))\mathrm{d}w\,\mathrm{d}r.$$

Moreover, again from the series representation already cited, we observe that

$$\overline{I_{2ir}(u)}=I_{-2ir}(u)\qquad(r,u\in\mathbb{R}),$$

therefore $\mathrm{Re}(I_{2ir}(v\sinh(w)))$ and $\mathrm{Im}(I_{2ir}(v\sinh(w)))$ are even and odd functions of r, respectively. Also, by the connection formula 10.27.2 of [48], which states

$$I_{-\nu}(Z)=I_\nu(Z)+\frac{2\sin(\nu\pi)}{\pi}K_\nu(Z)\qquad(\nu,Z\in\mathbb{C}),$$

we deduce the relation

$$\mathrm{Im}(I_{2ir}(u)) = -\frac{\sinh(2\pi r)}{\pi} K_{2ir}(u) \qquad (r, u \in \mathbb{R})\,.$$

Therefore we compute

$$\begin{aligned} g_\xi(\kappa, t, v) &= \frac{4}{\pi i} \int_\xi^\infty r \sinh(2\pi r) e^{-tr^2} \int_0^\infty e^{-v\cosh(w)} \left(\frac{\cosh(w)-1}{\cosh(w)+1}\right)^\kappa K_{2ir}(v \sinh(w))\,\mathrm{d}w\,\mathrm{d}r \\ &= \frac{4}{\pi i} \int_\xi^\infty r \sinh(2\pi r) e^{-tr^2} \int_0^\infty e^{-\sqrt{u^2+v^2}} \left(\frac{\sqrt{u^2+v^2}-v}{u}\right)^{2\kappa} K_{2ir}(u) \frac{\mathrm{d}u}{\sqrt{u^2+v^2}} \mathrm{d}r, \end{aligned}$$

where we performed the change of variable $u = v\sinh(w)$, with $\mathrm{d}w = \frac{\mathrm{d}u}{\sqrt{u^2+v^2}}$.
We are now in the hypothesis of formula 2.16.9.11 of [52] , which states

$$\begin{aligned} &\int_0^\infty e^{-p\sqrt{u^2+v^2}} \left(\frac{\sqrt{u^2+v^2}\pm v}{u}\right)^\mu K_\nu(cu) \frac{\mathrm{d}u}{\sqrt{u^2+v^2}} \\ &= \frac{1}{2cv} \Gamma\left(\frac{\nu \mp \mu + 1}{2}\right) \Gamma\left(\frac{-\nu \mp \mu + 1}{2}\right) W_{\pm\frac{\mu}{2},\frac{\nu}{2}}(v(p + \sqrt{p^2-c^2})) W_{\pm\frac{\mu}{2},\frac{\nu}{2}}(vp - \sqrt{p^2-c^2})) \\ &\qquad (\mathrm{Re}(v) > 0,\, \mathrm{Re}(p+c) > 0,\, |\mathrm{Re}(\nu)| < 1 \pm \mathrm{Re}(\mu))\,. \end{aligned}$$

Applying it to the w-integral, we find

$$g_\xi(\kappa, t, v) = \frac{2}{v\pi i} \int_\xi^\infty r \sinh(2\pi r) e^{-tr^2} \Gamma\left(\kappa + \frac{1}{2} + ir\right) \Gamma\left(\kappa + \frac{1}{2} - ir\right) W_{-\kappa, ir}(v)^2\,\mathrm{d}r. \tag{3.3.14}$$

By formulae (3.3.11) and (3.3.13) the right hand side of formula (3.3.14) is an absolutely and locally uniformly convergent integral whose integrand is holomorphic for $\mathrm{Im}(\kappa) \in (-\xi, \xi)$. It is thus holomorphic for κ ranging in the domain $\{\kappa \in \mathbb{C} \,|\, \mathrm{Im}(\kappa) \in (-\xi, \xi)\}$. By uniqueness of the analytic continuation, formula (3.3.14) holds in this more extended domain. In particular, if $\kappa \in \mathbb{Z}$ the relation holds for every $\xi > 0$. Since the integrand goes to zero for $r \to 0$, we can take limit for $\xi \to 0$ and conclude

$$\widehat{f}_k(t, \pm v) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi^2 v} \int_0^\infty r \sinh(2\pi r) e^{-tr^2} \left|\Gamma\left(\pm k + \frac{1}{2} + ir\right)\right|^2 W_{\mp k, ir}(v)^2\,\mathrm{d}r.$$

This completes the proof of the lemma. □

Lemma 3.3.2. *For $v = 0$ the continuous coefficients have the value*

$$\widehat{f}_k(t, 0) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\sqrt{\pi t}}.$$

Proof. We define the perturbed terms $\hat{f}_k(t,0,\eta)$ with a slightly different η-regularization than what we did in the previous lemma. Namely, we assume $0<\eta<\frac{1}{2}$ and we multiply the inner integral by $|w|^{-2\eta}$ instead of $(1+w^2)^{-\eta}$. Thus, we define

$$\widehat{f}_k(t,0,\eta) := \frac{(-1)^k e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi}\int_0^\infty r\tanh(\pi r)e^{-tr^2}\int_{-\infty}^{\infty}\left(\frac{w+i}{w-i}\right)^k\left(\frac{1}{w^2+1}\right)^{\frac{1}{2}+ir}|w|^{-2\eta}$$
$$\times\ {}_2F_1\left(-k+\frac{1}{2}+ir,k+\frac{1}{2}+ir,1;\frac{w^2}{w^2+1}\right)\mathrm{d}w\,\mathrm{d}r.$$

The limit

$$\lim_{\eta\to 0}\widehat{f}_k(t,0,\eta)=\widehat{f}_k(t,0)$$

holds by the same computation used to obtain formula (3.3.3).
We separately consider the w-integral occurring in the expression of $\widehat{f}_k(t,0,\eta)$. Observe that the integrand is an even function of w, except for the factor

$$\left(\frac{w+i}{w-i}\right)^k=\frac{(w+i)^{2k}}{(w^2+1)^k},$$

whose real and imaginary part are given by equations (3.2.7) and (3.2.8), respectively. Since the imaginary part is an odd function in w, we have

$$\widehat{f}_k(t,0,\eta)=\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\pi}\int_0^\infty r\tanh(\pi r)e^{-tr^2}\int_0^\infty\sum_{l=0}^{k}\binom{2k}{2l}\frac{w^{2l}(-1)^l}{(w^2+1)^k}\left(\frac{1}{w^2+1}\right)^{\frac{1}{2}+ir}w^{-2\eta}$$
$$\times\ {}_2F_1\left(-k+\frac{1}{2}+ir,k+\frac{1}{2}+ir,1;\frac{w^2}{w^2+1}\right)\mathrm{d}w\,\mathrm{d}r. \quad (3.3.15)$$

We now separately consider the quantity

$$I(r,\eta):=2\int_0^\infty\sum_{l=0}^{k}\binom{2k}{2l}\frac{(-1)^l w^{2(l-\eta)}}{(w^2+1)^{k+\frac{1}{2}+ir}}{}_2F_1\left(-k+\frac{1}{2}+ir,k+\frac{1}{2}+ir;1;\frac{w^2}{w^2+1}\right)\mathrm{d}w.$$

We apply the change of variable $v=\frac{w^2}{w^2+1}$, with $1-v=\frac{1}{w^2+1}$ and $\mathrm{d}w=\frac{(w^2+1)^2}{2w}\mathrm{d}v$, to obtain

$$I(r,\eta)=\sum_{l=0}^{k}\binom{2k}{2l}(-1)^l\int_0^1 v^{l-\eta-\frac{1}{2}}(1-v)^{k-l+\eta-1+ir}{}_2F_1\left(-k+\frac{1}{2}+ir,k+\frac{1}{2}+ir;1;v\right)\mathrm{d}v.$$

Formula 7.512 (3) of [30] states

$$\int_0^1 v^{\rho-1}(1-v)^{\beta-\rho-1}{}_2F_1(\alpha,\beta;\gamma;v)\mathrm{d}v=\frac{\Gamma(\gamma)\Gamma(\rho)\Gamma(\beta-\rho)\Gamma(\gamma-\alpha-\rho)}{\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\rho)}$$
$$(\mathrm{Re}(\rho)>0,\ \mathrm{Re}(\beta-\rho)>0,\ \mathrm{Re}(\gamma-\alpha-\rho)>0)\,.$$

Applying it, we find

$$I(r,\eta)=\sum_{l=0}^{k}\binom{2k}{2l}\frac{(-1)^l\Gamma(1)\Gamma\left(l-\eta+\frac{1}{2}\right)\Gamma\left(k-l+\eta+ir\right)\Gamma\left(k-l+\eta-ir\right)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-l+\eta+\frac{1}{2}\right)}.$$

Let us observe that, for $r>0$, the last expression is holomorphic in a neighborhood of $\eta=0$. Moreover, its asymptotics for r large can be bounded, using equation (3.3.6), as

$$|I(r,\eta)|\ll_k\cosh(\pi r).$$

We also consider the behavior for r small. If $\eta>0$ the quantity $I(r,\eta)$ has a finite limit for $r\to 0$, while if $\eta=0$ all the summands corresponding to $l\neq k$ have finite limit, and the summand corresponding to $l=k$ has the form

$$\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(ir\right)\Gamma\left(-ir\right)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}\right)}$$
$$=\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}\right)}\frac{\pi}{r\sinh(\pi r)},$$

where we used equation (3.3.9). Therefore the r-integral in (3.3.15) is absolutely convergent for any value of η in a neighborhood of 0, and we can apply Lebesgue's theorem to exchange the limit for $\eta\to 0^+$ and the r-integral. Thus, we find

$$\widehat{f}_{t,k}(0)$$
$$=\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi}\int_0^\infty r\tanh(\pi r)e^{-tr^2}\sum_{l=0}^{k}\binom{2k}{2l}\frac{(-1)^l\Gamma\left(l+\frac{1}{2}\right)\Gamma\left(k-l+ir\right)\Gamma\left(k-l-ir\right)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-l+\frac{1}{2}\right)}\mathrm{d}r.$$

Now we further simplify the sum occurring in the integral, which equals $I(r,0)$. We write the binomial in terms of Γ-functions

$$I(r,0)=\sum_{l=0}^{k}\frac{(-1)^l\Gamma(2k+1)\Gamma\left(l+\frac{1}{2}\right)\Gamma(k-l+ir)\Gamma(k-l-ir)}{\Gamma(2k-2l+1)\Gamma(2l+1)\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-l+\frac{1}{2}\right)}.$$

Applying the duplication formula (3.2.10) on the factors $\Gamma(2k+1)$, $\Gamma(2k-2l+1)$ and $\Gamma(2l+1)$ we obtain

$$I(r,0)=\sum_{l=0}^{k}\frac{(-1)^l\sqrt{\pi}\,\Gamma\left(k+\frac{1}{2}\right)\Gamma(k+1)\Gamma(k-l+ir)\Gamma(k-l-ir)}{\Gamma(k-l+1)\Gamma\left(k-l+\frac{1}{2}\right)\Gamma(l+1)\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-l+\frac{1}{2}\right)}.$$

We introduce the auxiliary parameter $m=k-l$, and we write the m-summand in terms of Pochhammer symbols. The Pochhammer symbol $(-k)_m$ is given, using the functional relation for the Γ-function, by the expression

$$(-k)_m=\frac{\Gamma(-k+m)}{\Gamma(-k)}=\frac{(-1)^m\Gamma(k+1)}{\Gamma(k-m+1)\Gamma(m+1)}.$$

Recalling $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we obtain

$$I(r,0) = \frac{(-1)^k \Gamma\left(k+\frac{1}{2}\right)\Gamma(ir)\Gamma(-ir)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}\right)} \sum_{m=0}^{k} \frac{(-k)_m\,(ir)_m\,(-ir)_m}{\left(\frac{1}{2}\right)_m\left(-k+\frac{1}{2}\right)_m}.$$

The m-sum is a generalized hypergeometric function

$$\sum_{m=0}^{k} \frac{(-k)_m\,(ir)_m\,(-ir)_m}{\left(\frac{1}{2}\right)_m\left(-k+\frac{1}{2}\right)_m} = {}_3F_2\left(-k, ir, -ir; \frac{1}{2}, -k+\frac{1}{2}; 1\right),$$

which is in the form prescribed by the Saalschütz theorem, cited as equation (3.2.11). Applying it, we find

$$\begin{aligned} I(r,0) &= \frac{(-1)^k \Gamma\left(k+\frac{1}{2}\right)\Gamma(ir)\Gamma(-ir)}{\Gamma\left(k+\frac{1}{2}+ir\right)\Gamma\left(k+\frac{1}{2}-ir\right)\Gamma\left(-k+\frac{1}{2}\right)} \frac{\left(\frac{1}{2}-ir\right)_k\left(\frac{1}{2}+ir\right)_k}{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k} \\ &= \frac{\Gamma(ir)\Gamma(-ir)}{\Gamma\left(\frac{1}{2}-ir\right)\Gamma\left(\frac{1}{2}+ir\right)} \frac{(-1)^k\pi}{\Gamma\left(-k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)}. \end{aligned}$$

Using formula (3.3.9) and the reflection formula 5.5.3 of [48], which states

$$\Gamma(Z)\Gamma(1-Z) = \frac{\pi}{\sin(\pi Z)} \qquad (Z \in \mathbb{C},\, Z \notin \mathbb{Z}),$$

we obtain the simplified expression

$$I(r,0) = \frac{1}{r\tanh(\pi r)}.$$

Therefore, we finally have

$$\widehat{f_{t,k}}(0) = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{2\pi} \int_0^\infty e^{-tr^2}\mathrm{d}r.$$

Computing the remaining integral proves the lemma. □

3.4 Formula for the heat kernel on the model cusp

In this section we state and prove, combining the lemmata of the last sections, a formula for the heat kernel on the model cusp defined in formula (3.1.1).

Theorem 3.4.1. *The on-diagonal heat kernel of weight k on the model cusp has the expression*

$$\begin{aligned} K_k^\infty(t;z,z) = \sum_{n=1}^\infty \frac{1}{4\pi n} \sum_{j=0}^{k-1} \frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\,\Gamma(j+1)} W_{k,k-j-\frac{1}{2}}(4\pi ny)^2 + y\,\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}} \\ + \sum_{n=1}^\infty \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi^3 n} \int_0^\infty r\sinh(2\pi r)e^{-tr^2} \sum_{\kappa=\pm k} \left|\Gamma\left(\kappa+\frac{1}{2}+ir\right)\right|^2 W_{-\kappa,ir}(4\pi ny)^2\,\mathrm{d}r. \end{aligned}$$

Proof. Formula 13.14.21 of [48] gives the asymptotic expansion

$$W_{\kappa,\mu}(Z) \sim e^{-\frac{Z}{2}} Z^{\kappa} \qquad \left(|\arg(Z)| < \frac{3\pi}{2}, |Z| \to \infty \right). \tag{3.4.1}$$

Combining it with equation (3.2.1), it shows the asymptotic behavior

$$\widehat{h}_k(t,-v) = O_{t,k}\left(v^{2k-1} e^{-v}\right) \qquad (v \to \infty)\,.$$

On the other hand the integrand appearing in formula (3.3.1) is a non-negative function. Combining this expression with the bound (3.3.11) and the relation (3.3.13) we obtain, tor any $t > 0$, the estimates

$$\widehat{f}_k(t,v) = O_{t,k}\left(\frac{W_{-k,0}(v)^2}{v}\right) \qquad (v \to \infty)\,,$$

$$\widehat{f}_k(t,-v) = O_{t,k}\left(v^{2k-1} W_{0,0}(v)^2 + v^{2k} W_{0,0}(v) W_{-1,0}(v) + v^{2k+1} W_{-1,0}(v)^2\right) \qquad (v \to \infty)\,.$$

Applying again equation (3.4.1), we obtain

$$\widehat{f}_k(t,\pm v) = O_{t,k}\left(v^{\mp 2k-1} e^{-v}\right) \qquad (v \to \infty)\,. \tag{3.4.2}$$

Thus, the right hand side of the formal relation (3.1.6) is absolutely convergent, and Poisson summation formula, theorem 4.2.8 in [51], implies that it is an equality. Replacing the explicit values for the coefficients $\widehat{h}_k(t, 4\pi ny)$ and $\widehat{f}_k(t, 4\pi ny)$ obtained in lemmata 3.2.1, 3.2.2, 3.3.1 and 3.3.2 completes the proof of the theorem. □

We now observe the connection between theorem (3.4.1) and the formula of Müller [46].

Remark 3.4.2. Setting $k = 0$ in the formula of theorem 3.4.1 we obtain

$$\begin{aligned} K_0^{\infty}(t;z,z) &= \frac{y\, e^{-\frac{t}{4}}}{\sqrt{4\pi t}} + \frac{e^{-\frac{t}{4}}}{2\pi^3} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} r \sinh(2\pi r) e^{-tr^2} \left| \Gamma\left(\frac{1}{2} + ir\right) \right|^2 W_{0,ir}(4\pi ny)^2 \, \mathrm{d}r \\ &= \frac{y\, e^{-\frac{t}{4}}}{\sqrt{4\pi t}} + \frac{4y\, e^{-\frac{t}{4}}}{\pi^2} \sum_{n=1}^{\infty} \int_0^{\infty} r \sinh(\pi r) e^{-tr^2} K_{ir}(2\pi ny)^2 \, \mathrm{d}r, \end{aligned}$$

where we applied equations (3.3.9), the duplication formula of the sinh-function and formula 13.18.9 of [48], which states

$$K_{\nu}(Z) = \sqrt{\frac{\pi}{2Z}}\, W_{0,\nu}(2Z) \qquad (\nu, Z \in \mathbb{C})\,.$$

Now, we observe that an orthonormal basis of eigenfunctions for the Laplacian on functions on the circle S^1 parametrized by $u \in [0,1)$ is

$$\{1\} \cup \left\{ \sqrt{2}\sin(2\pi nu),\ \sqrt{2}\cos(2\pi nu) \right\}_{n \in \mathbb{N}_{\geq 1}},$$

with associated eigenvalues 0 and $(2\pi n)^2$, respectively. We thus recover formula (2.29) of [46] in the case of the on-diagonal heat kernel on $\mathbb{R}_{>0} \times S^1 \simeq \mathcal{F}_{\infty}$ equipped with the cusp metric, i.e., the hyperbolic metric on $\mathcal{F}_{\infty}$.

This remark suggests that the result of Müller can be generalized to the heat kernel associated to the Laplacian on $(k, 0)$-tensors on an arbitrary closed manifold. Specifically, we suspect that the proof of Müller can be adapted to this generalized setting using theorem A.2 in place of the Kontorovich–Lebedev inversion formula.

Chapter 4

A regularized arithmetic Riemann–Roch theorem

In this chapter we define a regularization of the hyperbolic metric in an ϵ-neighborhood of the cusps, for a small parameter $0 < \epsilon \ll 1$. The line bundle of cusp forms equipped with this regularized metric is a smooth hermitian line bundle defined on a compact Riemann surface equipped with a Kähler metric such that the arithmetic Riemann–Roch theorem applies. We then examine the behavior of the terms appearing in the theorem through metric degeneration, i.e., in the limit for $\epsilon \to 0$.

4.1 The metric degeneration

Let $f\colon \mathcal{X} \to \mathcal{S}$ be an arithmetic surface over $\mathcal{S}$ such that $X = \mathcal{X}_{\mathbb{C}} \simeq X(\Gamma)$ for Γ a cofinite torsion-free and discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We further assume that X is equipped with a hermitian metric that makes it isometric to $X(\Gamma)$ equipped with the hyperbolic metric. By abuse of notation, we identify the Riemann surfaces X with $X(\Gamma)$.
Now, for $k \geq 0$, let $\overline{\mathcal{S}}_{k+1}$ be a hermitian line bundle on $\mathcal{X}$ such that the induced complex hermitian line bundle is isometric to the line bundle of cusp forms $\overline{S}_{k+1}$ on X. By abuse of notation, we identify $\overline{\mathcal{S}}_{k+1,\mathbb{C}}$ with $\overline{S}_{k+1}$.

We now regularize the Petersson metric and the hyperbolic metric in a neighborhood of the cusps. Since it will be needed in section 4.2, we give a precise analytic description of the regularized metric. Let $0 < \epsilon \ll 1$ be a small parameter and $\phi(\epsilon)$ a continuous function of ϵ such that $\epsilon - \epsilon^2 \leq \phi(\epsilon) < \epsilon$, e.g., $\phi(\epsilon) = \epsilon - \epsilon^2$. Let $h(v)$ be the smooth transition function, defined by

$$h(v) := \frac{a(v)}{a(v) + a(1-v)} \qquad (v \in \mathbb{R}),$$

where

$$a(v) := \begin{cases} 0, & v \leq 0, \\ e^{-\frac{1}{v}}, & v > 0, \end{cases} \qquad (v \in \mathbb{R}).$$

We then define the smooth transition function between $\phi(\epsilon)$ and ϵ by the formula

$$h_\epsilon(r) := h\left(\frac{r - \phi(\epsilon)}{\epsilon - \phi(\epsilon)}\right) = \begin{cases} 0, & r \leq \phi(\epsilon), \\ 1, & r \geq \epsilon, \end{cases} \qquad (r \in \mathbb{R}_{\geq 0}),$$

and we use it to interpolate the functions $r\left(-\frac{\log(r)}{2\pi}\right)^k$ and $\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^k$ and define

$$\rho_{k,\epsilon}(r) := h_\epsilon(r)\, r\left(-\frac{\log(r)}{2\pi}\right)^k + (1-h_\epsilon(r))\,\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^k \qquad (r\in\mathbb{R}_{\geq 0})\,.$$

Let us observe that

$$\rho_{k,\epsilon}(r) = \begin{cases} \phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^k, & 0\leq r\leq\phi(\epsilon),\\ \text{smooth, monotone increasing}, & \phi(\epsilon)\leq r\leq\epsilon,\\ r\left(-\frac{\log(r)}{2\pi}\right)^k, & r\geq\epsilon. \end{cases}$$

As in the proof of proposition 2.5.4, we fix $s_\Gamma\in\mathbb{R}_{>0}$ such that $B_{s_\Gamma}(P_j)\cap B_{s_\Gamma}(P_h)=\emptyset$ for $j\neq h$ and $-\log(s_\Gamma)>2\pi$, and $q_j := e^{2\pi i\sigma_j(z)}$ is a local coordinate in each neighborhood $B_{s_\Gamma}(P_j)$. We further restrict to $\epsilon < s_\Gamma$ such that S_{k+1} trivializes in the neighborhood $B_\epsilon(i\infty)$.

Definition 4.1.1. Let $g\in A^{0,0}(X,S_{k+1})$ be a smooth cusp form of weight $2(k+1)$. Its *ϵ-regularized Petersson norm* is given by the assignments

$$\|g(q_j)\|_{\overline{S}_{k+1,\epsilon}} := \|g(q_j)\|_{\overline{S}_{k+1}}\,\frac{\rho_{k+1,\epsilon}(|q_j|)}{|q_j|\left(-\frac{\log|q_j|)}{2\pi}\right)^{k+1}} \qquad (q_j\in B_{s_\Gamma}(P_j),\, j\in\{1,...,p\})\,,$$

and

$$\|g(z)\|_{\overline{S}_{k+1,\epsilon}} := \|g(z)\|_{\overline{S}_{k+1}} \qquad \left(z\in X\setminus\bigcup_{j=1}^{p} B_{s_\Gamma}(P_j)\right).$$

We remark that this metric is well-defined, since we have an isomorphism

$$A^{0,0}(B_\epsilon(i\infty),S_{k+1}) \xrightarrow{\simeq} A^{0,0}(B_\epsilon(i\infty),\mathcal{O}_X).$$

The ϵ-regularized Petersson norm defines a smooth hermitian metric on S_{k+1}. We denote by $\overline{\mathcal{S}}_{k+1,\epsilon}$ the resulting smooth hermitian line bundle, and by $\overline{S}_{k+1,\epsilon}$ its complex part.

Definition 4.1.2. The *ϵ-regularized hyperbolic metric* $\|\cdot\|_{\mathrm{hyp},\epsilon}$ is the smooth hermitian metric on ω_X making the isomorphism $S_1\simeq\omega_X$ an isometry.

We denote by $\overline{\omega}_{\mathcal{X},\epsilon}$ the relative dualizing sheaf equipped with the ϵ-regularized hyperbolic metric. The ϵ-regularized hyperbolic norm is given by the expressions

$$\|\mathrm{d}q_j\|_{\mathrm{hyp},\epsilon} = 2\pi\rho_{1,\epsilon}(|q_j|) \qquad (q_j\in B_{s_\Gamma}(P_j),\, j\in\{1,...,p\})\,,$$

and

$$\|\mathrm{d}z\|_{\mathrm{hyp},\epsilon} = \|\mathrm{d}z\|_{\mathrm{hyp}} \qquad \left(z\in X\setminus\bigcup_{j=1}^{p} B_{s_\Gamma}(P_j)\right).$$

This metric induces a smooth Kähler metric on X, also called the ϵ-regularized hyperbolic metric, whose volume form is given by the expressions

$$\mu_\epsilon(q_j) = \frac{i}{2}\frac{\mathrm{d}q_j \wedge \mathrm{d}\bar{q}_j}{4\pi^2\, \rho_{1,\epsilon}(|q_j|)^2} \qquad (q_j \in B_{s_\Gamma}(P_j),\, j \in \{1,...,p\})\,,$$

and

$$\mu_{\mathrm{hyp},\epsilon}(z) = \mu_{\mathrm{hyp}}(z) \qquad \left(z \in X \setminus \bigcup_{j=1}^{p} B_{s_\Gamma}(P_j)\right).$$

Notation 4.1.3. We write

$$\mu_\epsilon(z) := \mu_{\mathrm{hyp},\epsilon}(z) \qquad (z \in X)\,,$$

and

$$\mathrm{vol}_\epsilon(U) := \int_U \mu_\epsilon(z) \qquad (U \subseteq X)\,.$$

Moreover, by abuse of notation, we denote by $\langle\cdot,\cdot\rangle_{\mathrm{Pet},\epsilon}$ the L^2-metrics on $A^{0,0}(X, S_{k+1})$ and $A^{0,1}(X, S_{k+1})$ induced by the ϵ-regularized Petersson metric on S_{k+1} and the ϵ-regularized hyperbolic metric on X according to definition 1.4.5.

Finally, we define the smooth complex hermitian line bundle $\overline{M}_{k,\epsilon}$ via the isometric isomorphism

$$\overline{M}_{k,\epsilon} := \overline{S}_{k+1,\epsilon} \otimes \overline{S}^{\vee}_{1,\epsilon}.$$

If $h \in A^{0,0}(X, M_k)$ is a smooth modular form of weight $2k$, its ϵ-regularized norm is given by

$$\|h(q_j)\|_{\overline{M}_{k,\epsilon}} = \|h(q_j)\|_{\overline{M}_k} \frac{\rho_{k+1,\epsilon}(|q_j|)}{\rho_{1,\epsilon}(|q_j|)\left(-\frac{\log|q_j|}{2\pi}\right)^k} \qquad (q_j \in B_{s_\Gamma}(P_j),\, j \in \{1,...,p\})\,,$$

and

$$\|h(z)\|_{\overline{M}_{k,\epsilon}} = \|h(z)\|_{\overline{M}_k} \qquad \left(z \in X \setminus \bigcup_{j=1}^{p} B_{s_\Gamma}(P_j)\right).$$

We can now apply the smooth arithmetic Riemann–Roch theorem, in the version stated in corollary 1.5.5, to the line bundle $\overline{\mathcal{S}}_{k+1,\epsilon}$ on the arithmetic surface $f\colon \mathcal{X}\to\mathcal{S}$, whose complex fiber X is equipped with the ϵ-regularized hyperbolic metric. We obtain

$$\begin{aligned}\widehat{\deg}\,\widehat{\mathrm{c}}_1(\lambda(\overline{\mathcal{S}}_{k+1,\epsilon})_Q) = {} & \frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\mathcal{S}}_{k+1,\epsilon} - 6\overline{\mathcal{S}}_{k+1,\epsilon}, \overline{\omega}_{\mathcal{X},\epsilon} + \overline{\omega}_{\mathcal{X},\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}\right) - \delta_f \\ & + \frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1(\overline{\omega}_{X,\epsilon}).\end{aligned} \tag{4.1.1}$$

And, by definition 1.1.10 of the arithmetic degree and definition 1.4.18 of the Quillen metric, the left hand side decomposes as

$$\widehat{\deg}\,\widehat{\mathrm{c}}_1(\lambda(\overline{\mathcal{S}}_{k+1,\epsilon})_Q) = \widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet},\epsilon}\right) + \widehat{\deg}\left(\det \mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet},\epsilon}\right) + \frac{1}{2}\log\left(\det{}'\left(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\right)\right).$$

We call the quantity $\det{}'\left(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\right)$ the smoothened determinant, not to be confused with the regularized determinant $\det^{*}_{\Gamma}\left(\Delta^{1}_{\overline{S}_{k+1}}\right)$. We can now examine the behavior for $\epsilon\to 0$ of the terms appearing in equation (4.1.1), starting from those that admit an elementary degeneration. First, the contribution of the singular fibers δ_f is independent of ϵ. By the Gauss–Bonnet theorem the same holds for the topological term

$$\frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1(\overline{\omega}_{X,\epsilon}) = \left(2\zeta'(-1)+\zeta(-1)\right)(g-1).$$

Regarding the volume of cohomology we observe that the limit for $\epsilon\to 0$ only involves the fiber at the archimedean place, and, as in [59, page 122], the complex fiber of the determinant of cohomology is given by

$$\lambda\left(\mathcal{S}_{k+1}\right)_{\mathbb{C}} = \det \mathrm{H}^0(X,S_{k+1})\otimes \det \mathrm{H}^1(X,S_{k+1})^{\vee}.$$

Using this we now show that

$$\lim_{\epsilon\to 0}\widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet},\epsilon}\right) = \widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet}}\right). \tag{4.1.2}$$

By definition 1.1.10 of the arithmetic degree, it is enough to show

$$\lim_{\epsilon\to 0}\log\left(\det\left(\langle g_j,g_h\rangle_{\mathrm{Pet},\epsilon}\right)_{j,h=1,\ldots,m_{k+1}}\right) = \log\left(\det\left(\langle g_j,g_h\rangle_{\mathrm{Pet}}\right)_{j,h=1,\ldots,m_{k+1}}\right),$$

for a basis $\{g_j\}_{j=1}^{m_{k+1}}$ of $\mathrm{H}^0(X,S_{k+1})$. The limit

$$\lim_{\epsilon\to 0}\langle g_j,g_h\rangle_{\mathrm{Pet},\epsilon} = \langle g_j,g_h\rangle_{\mathrm{Pet}} \qquad \left(g_j,g_h\in \mathrm{H}^0(X,S_{k+1})\right)$$

is immediate by the definition of the ϵ-regularized Petersson metric. For the remaining term we observe that, by Serre's duality, we have

$$\mathrm{H}^1(X,S_{k+1})^{\vee}\simeq \mathrm{H}^0(X,M_{-k}) = \begin{cases}\mathbb{C}, & k=0,\\ \{0\}, & k\geq 1.\end{cases}$$

Thus, we have the equality $\det \mathrm{H}^0(X,M_{-k})=\mathbb{C}$ for any k, and, considering its section 1, we compute

$$\log\left(\|1\|^2_{\mathrm{Pet},\epsilon}\right) = \log\left(\int_X 1^2\,\mu_\epsilon(z)\right) = \log\left(\mathrm{vol}_\epsilon(X)\right)\xrightarrow[\epsilon\to 0]{}\log\left(\mathrm{vol}_{\mathrm{hyp}}(X)\right).$$

Specifically, we have

$$\lim_{\epsilon\to 0}\widehat{\deg}\left(\det \mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet},\epsilon}\right) = \widehat{\deg}\left(\det \mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet}}\right). \tag{4.1.3}$$

It remains to consider the degeneration of the smoothened determinant on the left hand side and of the arithmetic intersection numbers on the right hand side of formula (4.1.1).

4.2 Degeneration of the arithmetic intersection numbers

In this section we establish the asymptotic expansion for $\epsilon \to 0$ of the intersection numbers

$$\frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\mathcal{S}}_{k+1,\epsilon} - 6\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon} + \overline{\omega}_{\mathcal{X},\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}\right) \tag{4.2.1}$$
$$= \frac{p}{12}\log(\epsilon) + p\left(\frac{k}{2}+\frac{1}{6}\right)\log(-\log(\epsilon)) + O_{\mathcal{X},k}(1).$$

Before proving formula (4.2.1), we need to compute the asymptotic behavior of some integrals later occurring in the proof. We will do so by applying the following technical lemma.

Lemma 4.2.1. *Let $a(v), b(v)$ be smooth real-valued functions such that $0 < a(v) < b(v)$ for $v > 0$. Moreover, let the following conditions be satisfied:*

(1) For any $v > 0$, the real-valued functions $f(v,r)$ and $g(v,r)$ are smooth for $r \in [a(v), b(v)]$.

(2) For any $v > 0$ and $r \in [a(v), b(v)]$, the function $f(v,r)$ has constant sign.

(3) For any $v > 0$, the function $\frac{\mathrm{d}}{\mathrm{d}r}g(v,r)$ changes sign finitely many times for $r \in [a(v), b(v)]$, and the number of sign changes is independent of v.

(4) The limit

$$\lim_{v\to 0}\left(\max_{r\in[a(v),b(v)]}\{f(v,r)\} - \min_{r\in[a(v),b(v)]}\{f(v,r)\}\right)\cdot \max_{r\in[a(v),b(v)]}|g(v,r)| = 0$$

holds.

Then, we have the asymptotic expansion

$$\int_{a(v)}^{b(v)} f(v,r)\,\frac{\mathrm{d}}{\mathrm{d}r}g(v,r)\mathrm{d}r = f(v,b(v))\left(g(v,b(v)) - g(v,a(v))\right) + o(1) \qquad (v\to 0)\,.$$

Proof. To prove the lemma we give an upper and a lower bound for the integral. To simplify the exposition we assume $f(v,r) \geq 0$, and that there exists a function $a(v) < c(v) < b(v)$ such that $\frac{\mathrm{d}}{\mathrm{d}r}g(v,r) \geq 0$ for $r \leq c(v)$ and $\frac{\mathrm{d}}{\mathrm{d}r}g(v,r) \leq 0$ for $r \geq c(v)$. Then, we compute

$$\int_{a(v)}^{b(v)} f(v,r)\,\frac{\mathrm{d}}{\mathrm{d}r}g(v,r)\mathrm{d}r$$
$$\leq \max_{r\in[a(v),c(v)]}\{f(v,r)\}\int_{a(v)}^{c(v)} \frac{\mathrm{d}}{\mathrm{d}r}g(v,r)\mathrm{d}r + \min_{r\in[c(v),b(v)]}\{f(v,r)\}\int_{c(v)}^{b(v)} \frac{\mathrm{d}}{\mathrm{d}r}g(v,r)\mathrm{d}r$$

$$= \max_{r\in[a(v),b(v)]}\{f(v,r)\}\left(g(v,c(v))-g(v,a(v))\right)+\min_{r\in[a(v),b(v)]}\{f(v,r)\}\left(g(v,b(v))-g(v,c(v))\right)$$
$$= \max_{r\in[a(v),b(v)]}\{f(v,r)\}\left(g(v,b(v))-g(v,a(v))\right)$$
$$+\left(\max_{r\in[a(v),b(v)]}\{f(v,r)\}-\min_{r\in[a(v),b(v)]}\{f(v,r)\}\right)\left(g(v,c(v))-g(v,b(v))\right).$$

We apply the key requirement (4) to ensure that the last term is $o(1)$, and that in the first term we can replace $\max_{r\in[a(v),b(v)]}\{f(v,r)\}$ by $f(v,b(v))$. Thus, we find

$$\int_{a(v)}^{b(v)} f(v,r)\,\frac{\mathrm{d}}{\mathrm{d}r}g(v,r)\mathrm{d}r \leq f(v,b(v))\left(g(v,b(v))-g(v,a(v))\right)+o(1) \qquad (v\to 0).$$

We similarly prove a corresponding lower bound. □

In the next lemma we state the asymptotic expansion of four notable integrals that will later occur in the computation of the arithmetic intersection numbers.

Lemma 4.2.2. *For $\epsilon\to 0$ we have the asymptotic expansions*

$$\int_{\phi(\epsilon)}^{\epsilon} r\log\frac{\rho_{k+1,\epsilon}(r)}{\rho_{1,\epsilon}(r)}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{k+1,\epsilon}(r)\mathrm{d}r = k\log(-\log(\epsilon))-k\log(2\pi)+o(1), \tag{4.2.2}$$

$$\int_{\phi(\epsilon)}^{\epsilon} \log\frac{\rho_{k+1,\epsilon}(r)}{\rho_{1,\epsilon}(r)}\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{k+1,\epsilon}(r)\mathrm{d}r = o(1), \tag{4.2.3}$$

$$\int_{\phi(\epsilon)}^{\epsilon} r\,\log\rho_{1,\epsilon}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{1,\epsilon}(r)\mathrm{d}r = \log(\epsilon)+\log(-\log(\epsilon))-\log(2\pi)+1+o(1), \tag{4.2.4}$$

$$\int_{\phi(\epsilon)}^{\epsilon} \log\rho_{1,\epsilon}(r)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{1,\epsilon}(r)\mathrm{d}r = o(1). \tag{4.2.5}$$

Proof. All the formulae occurring in the lemma are applications of lemma (4.2.1). Let $l\in\mathbb{N}$, we compute

$$\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^l-\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^l$$
$$=(\epsilon-\phi(\epsilon))\left(-\frac{\log(\epsilon)}{2\pi}\right)^l-\phi(\epsilon)\left(\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^l-\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right).$$

The left hand side of the last relation is positive, because the function $r\left(-\frac{\log(r)}{2\pi}\right)^l$ is monotone increasing for r small enough. Moreover

$$\phi(\epsilon)\left(\left(-\frac{\log\phi(\epsilon))}{2\pi}\right)^l-\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right)>0.$$

Therefore, we deduce

$$\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^l - \phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^l < (\epsilon-\phi(\epsilon))\left(-\frac{\log(\epsilon)}{2\pi}\right)^l. \tag{4.2.6}$$

Similary, we compute

$$\begin{aligned}
&\left|\phi(\epsilon)\log\left(\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^l\right) - \epsilon\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right)\right| \\
&\qquad = \epsilon\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right) - \phi(\epsilon)\log\left(\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^l\right) \\
&\qquad = (\epsilon-\phi(\epsilon))\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right) - l\phi(\epsilon)\log\left(\frac{-\log\phi(\epsilon)}{-\log(\epsilon)}\right) \\
&\qquad < (\epsilon-\phi(\epsilon))\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^l\right).
\end{aligned} \tag{4.2.7}$$

We now prove formulae (4.2.2) and (4.2.4). Let $n, m \in \mathbb{N}$, we consider the integral

$$\int_{\phi(\epsilon)}^{\epsilon} r\,\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{m,\epsilon}(r)\,\mathrm{d}r.$$

Our aim is to apply lemma 4.2.1 with the notation $v=\epsilon$, $r=r$, $a(\epsilon)=\phi(\epsilon)$, $b(\epsilon)=\epsilon$, $f(\epsilon,r)=r\log\rho_{n,\epsilon}(r)$ and $g(\epsilon,r)=\frac{\mathrm{d}}{\mathrm{d}r}(\log\rho_{m,\epsilon}(r))$. Requirements (1), (2) and (3) are satisfied by construction. We now verify the requirement (4), which is

$$\lim_{\epsilon\to 0}\left(\max_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\} - \min_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\}\right)\cdot\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right| = 0. \tag{4.2.8}$$

Since, for ϵ small enough, $r\log\rho_{n,\epsilon}(r)$ is negative and monotone decreasing in the interval $(\phi(\epsilon),\epsilon)$, we have

$$\max_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\} = \phi(\epsilon)\,\log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^n\right)$$

and

$$\min_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\} = \epsilon\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right).$$

Therefore, since $\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right) \ll_n -\frac{\log(\epsilon)}{2\pi}$, using formula (4.2.6) with $l=1$ and formula (4.2.7) with $l=n$, we find

$$\begin{aligned}
&\max_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\}-\min_{r\in[\phi(\epsilon),\epsilon]}\{r\log\rho_{n,\epsilon}(r)\}\\
&\qquad=\phi(\epsilon)\log\phi(\epsilon)-\epsilon\log(\epsilon)+\phi(\epsilon)\log\left(\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^n\right)-\epsilon\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right)\\
&\qquad=2\pi\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)-\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)\right)\\
&\qquad\quad+\phi(\epsilon)\log\left(\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^n\right)-\epsilon\log\left(\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right)\\
&\qquad\ll_n(\epsilon-\phi(\epsilon))\left(-\frac{\log(\epsilon)}{2\pi}\right).
\end{aligned}$$

On the other hand, differentiating the logarithm we obtain

$$\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right|\leq\frac{\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\rho_{m,\epsilon}(r)\right|}{\min_{r\in[\phi(\epsilon),\epsilon]}\{\rho_{m,\epsilon}(r)\}}.$$

Regarding the denominator of the last expression, since $\rho_{m,\epsilon}(r)$ is monotone increasing, we have

$$\min_{r\in[\phi(\epsilon),\epsilon]}\{\rho_{m,\epsilon}(r)\}=\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m.$$

While for the numerator we use the explicit expression of $\rho_{m,\epsilon}(r)$, and we find

$$\begin{aligned}
\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\rho_{m,\epsilon}(r)\right|&=\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}h_\epsilon(r)\left(r\left(-\frac{\log(r)}{2\pi}\right)^m-\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right)\right.\\
&\qquad\left.+h_\epsilon(r)\left(\left(-\frac{\log(r)}{2\pi}\right)^m-\frac{m}{2\pi}\left(-\frac{\log(r)}{2\pi}\right)^{m-1}\right)\right|\\
&\leq\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}h_\epsilon(r)\right|\cdot\max_{r\in[\phi(\epsilon),\epsilon]}\left|r\left(-\frac{\log(r)}{2\pi}\right)^m-\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right|\\
&\quad+\max_{r\in[\phi(\epsilon),\epsilon]}\left|\left(-\frac{\log(r)}{2\pi}\right)^m-\frac{m}{2\pi}\left(-\frac{\log(r)}{2\pi}\right)^{m-1}\right|.
\end{aligned}$$

Using the fact that the functions $r\left(-\frac{\log(r)}{2\pi}\right)^m$ and $\left(-\frac{\log(r)}{2\pi}\right)^m-\frac{m}{2\pi}\left(-\frac{\log(r)}{2\pi}\right)^{m-1}$ are monotone increasing and decreasing for r small enough, respectively, we have

$$\begin{aligned}
\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\rho_{m,\epsilon}(r)\right|\leq\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}h_\epsilon(r)\right|\cdot\left|\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^m-\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right|\\
+\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m-\frac{m}{2\pi}\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{m-1}.
\end{aligned}$$

By construction, we have

$$\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}h_\epsilon(r)\right| = \frac{\max\limits_{v\in[0,1]}\left|\frac{\mathrm{d}}{\mathrm{d}v}h(v)\right|}{\epsilon-\phi(\epsilon)} \ll \frac{1}{\epsilon-\phi(\epsilon)}.$$

Moreover, using formula (4.2.6) with $l=m$, we deduce

$$\left|\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^m - \phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right| < (\epsilon-\phi(\epsilon))\left(-\frac{\log(\epsilon)}{2\pi}\right)^m.$$

And, for ϵ small enough, we further observe the elementary inequality

$$\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m - \frac{m}{2\pi}\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{m-1} < \left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m.$$

Combining the pieces, we have

$$\max_{r\in[\phi(\epsilon),\epsilon]}\left|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right| \ll \frac{\frac{1}{\epsilon-\phi(\epsilon)}(\epsilon-\phi(\epsilon))\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m + \left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m}{\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m} = \frac{2}{\phi(\epsilon)}.$$

Substituting the bounds we found on the two factors of (4.2.8), we have to verify the limit

$$\lim_{\epsilon\to0}(\epsilon-\phi(\epsilon))\left(-\frac{\log(\epsilon)}{2\pi}\right)\frac{2}{\phi(\epsilon)} = 0,$$

which follows by the assumption $\epsilon-\phi(\epsilon)<\epsilon^2$. We can now apply lemma 4.2.1 and deduce that

$$\begin{aligned}\int\limits_{\phi(\epsilon)}^{\epsilon} & r\,\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{m,\epsilon}(r)\mathrm{d}r \\ &= \epsilon\,\log\rho_{n,\epsilon}(\epsilon)\left(\left(\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right)_{r=\epsilon} - \left(\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right)_{r=\phi(\epsilon)}\right) + o(1) \qquad (\epsilon\to0).\end{aligned}$$

By construction of the functions $\rho_{n,\epsilon}(r)$ and $\rho_{m,\epsilon}(r)$, we have

$$\epsilon\log\rho_{n,\epsilon}(\epsilon) = \epsilon\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right),$$

the relation $\left(\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right)_{r=\phi(\epsilon)}=0$, and

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\right)_{r=\epsilon} = \left(\frac{\mathrm{d}}{\mathrm{d}r}\log\left(r\left(-\frac{\log(r)}{2\pi}\right)^m\right)\right)_{r=\epsilon} = \frac{1}{\epsilon}+\frac{m}{\epsilon\log(\epsilon)}.$$

Therefore, we find the asymptotic expansion

$$\begin{aligned}\int\limits_{\phi(\epsilon)}^{\epsilon} r\,\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{m,\epsilon}(r)\mathrm{d}r &= \epsilon\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right)\left(\frac{1}{\epsilon}+\frac{m}{\epsilon\log(\epsilon)}\right)+o(1) \\ &= \log(\epsilon)+n\log(-\log(\epsilon))-n\log(2\pi)+m+o(1) \qquad (\epsilon\to0).\end{aligned}$$

Setting $n = m = 1$ in the last expression we prove formula (4.2.4), and subtracting the case $n = 1$ and $m = k+1$ from the case $n = m = k+1$ we prove formula (4.2.2). We now proceed similarly to prove formulae (4.2.3) and (4.2.5). For $l \in \mathbb{N}$ we compute

$$\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^{l}\right) - \log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{l}\right) = \log\left(\frac{\epsilon}{\phi(\epsilon)}\right) + l\log\left(\frac{-\log(\epsilon)}{-\log\phi(\epsilon)}\right),$$

and we observe that the left hand side is positive because the function $\log\left(r\left(-\frac{\log(r)}{2\pi}\right)^{l}\right)$ is monotone increasing for r small enough, the first summand on the right hand side is positive because $\epsilon > \phi(\epsilon)$ and the second summand on the right hand side is negative because $0 < -\log(\epsilon) < -\log\phi(\epsilon)$. Further using the inequality $\log(1+u) < u$ with $u = \frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)}$ we conclude

$$\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^{l}\right) - \log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{l}\right) < \log\left(\frac{\epsilon}{\phi(\epsilon)}\right) < \frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)}. \tag{4.2.9}$$

We now consider the integral

$$\int_{\phi(\epsilon)}^{\epsilon} \log\rho_{n,\epsilon}(r)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\mathrm{d}r.$$

We apply lemma 4.2.1 with the notation $v = \epsilon$, $r = r$, $a(\epsilon) = \phi(\epsilon)$, $b(\epsilon) = \epsilon$, $f(\epsilon, r) = \log\rho_{n,\epsilon}(r)$ and $g(\epsilon, r) = \log\rho_{m,\epsilon}(r)$. Requirements (1), (2) and (3) are trivially satisfied. We now verify requirement (4), which is

$$\lim_{\epsilon\to 0}\left(\max_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\} - \min_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\}\right)\cdot\max_{r\in[\phi(\epsilon),\epsilon]}|\log\rho_{m,\epsilon}(r)| = 0. \tag{4.2.10}$$

Since $\log\rho_{n,\epsilon}(r)$ is negative and monotone increasing in the interval $(\phi(\epsilon), \epsilon)$, we have

$$\max_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\} = \log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^{n}\right),$$

and

$$\min_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\} = \log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{n}\right).$$

Therefore, applying the inequality (4.2.9) with $l = n$, we find

$$\max_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\} - \min_{r\in[\phi(\epsilon),\epsilon]}\{\log\rho_{n,\epsilon}(r)\} < \frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)}.$$

On the other hand, using once more the fact that the function $\log\rho_{m,\epsilon}(r)$ is negative and monotone increasing in the interval $(\phi(\epsilon), \epsilon)$, we have

$$\max_{r\in[\phi(\epsilon),\epsilon]}|\log\rho_{m,\epsilon}(r)| = -\log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{m}\right).$$

Therefore the limit (4.2.10) reduces to

$$\lim_{\epsilon\to 0}\frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)}\log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right)=0,$$

which holds because of the assumption $\epsilon-\phi(\epsilon)<\epsilon^2$. We can now apply lemma 4.2.1 and deduce the asymptotic expansion

$$\int\limits_{\phi(\epsilon)}^{\epsilon}\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\mathrm{d}r=\log\rho_{n,\epsilon}(\epsilon)\left(\log\rho_{m,\epsilon}(\epsilon)-\log\rho_{m,\epsilon}(\phi(\epsilon))\right)+o(1)\qquad(\epsilon\to 0)\,.$$

By construction of the functions $\rho_{n,\epsilon}(r)$ and $\rho_{m,\epsilon}(r)$, we have

$$\log(\rho_{n,\epsilon}(\epsilon))=\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right),$$

and

$$\begin{aligned}\log\rho_{m,\epsilon}(\epsilon)-\log\rho_{m,\epsilon}(\phi(\epsilon))&=\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^m\right)-\log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^m\right)\\&<\frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)},\end{aligned}$$

where the last inequality is obtained applying equation (4.2.9) with $l=m$. Therefore, using the hypothesis $\epsilon-\phi(\epsilon)<\epsilon^2$, we find the asymptotic expansion

$$\left|\int\limits_{\phi(\epsilon)}^{\epsilon}\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{m,\epsilon}(r)\mathrm{d}r\right|<\left|\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^n\right)\frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)}\right|+o(1)=o(1)\qquad(\epsilon\to 0)\,.$$

Since the last asymptotics is valid for any $n,m\in\mathbb{N}$, the statements of formulae (4.2.3) and (4.2.5) follow.

□

We now compute the asymptotic expansion of two more notable integrals.

Lemma 4.2.3. *Let $a\in\mathbb{R}_{>0}$, $n\in\mathbb{N}$ and $\varphi\big(re^{i\theta}\big)$ be a non-zero holomorphic function defined on the ball $B_a(0)$ with polar coordinates r and θ. Let $\epsilon<a$, then we have the asymptotic expansions*

$$\int\limits_0^{2\pi}\int\limits_{\phi(\epsilon)}^{\epsilon}\log\left|\varphi\big(re^{i\theta}\big)\right|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}=o(1)\qquad(\epsilon\to 0)\,,\tag{4.2.11}$$

and

$$\int\limits_0^{2\pi}\int\limits_{\phi(\epsilon)}^{\epsilon}r\log\left|\varphi\big(re^{i\theta}\big)\right|\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{n,\epsilon}(r)\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}=\log|\varphi(0)|+o(1)\qquad(\epsilon\to 0)\,.\tag{4.2.12}$$

Proof. Regarding the first estimate, we bound the r-integral by its absolute value, obtaining

$$\left|\int\limits_{\phi(\epsilon)}^{\epsilon} \log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\mathrm{d}r\right| \leq \int\limits_{\phi(\epsilon)}^{\epsilon} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \left|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\right| \mathrm{d}r.$$

By assumption, the quantity

$$\max_{re^{i\theta}\in B_a(0)} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right|$$

is well-defined and finite. Moreover, since $\log\rho_{n,\epsilon}(r)$ is monotone increasing, we have

$$\begin{aligned}
&\int\limits_{\phi(\epsilon)}^{\epsilon} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \left|\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\right| \mathrm{d}r \\
&\qquad \leq \max_{re^{i\theta}\in B_a(0)} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \int\limits_{\phi(\epsilon)}^{\epsilon} \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\mathrm{d}r \\
&\qquad = \max_{re^{i\theta}\in B_a(0)} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \left(\log\rho_{n,\epsilon}(\epsilon) - \log\rho_{n,\epsilon}(\phi(\epsilon))\right) \\
&\qquad = \max_{re^{i\theta}\in B_a(0)} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \left(\log\left(\epsilon\left(-\frac{\log(\epsilon)}{2\pi}\right)^{n}\right) - \log\left(\phi(\epsilon)\left(-\frac{\log\phi(\epsilon)}{2\pi}\right)^{n}\right)\right) \\
&\qquad < \max_{re^{i\theta}\in B_a(0)} \left|\log\left|\varphi\left(re^{i\theta}\right)\right|\right| \frac{\epsilon-\phi(\epsilon)}{\phi(\epsilon)},
\end{aligned}$$

where the last inequality follows by equation (4.2.9) with $l = n$. Since $\epsilon - \phi(\epsilon) < \epsilon^2$, this completes the proof of the estimate (4.2.11). For the second asymptotic expansion, we integrate the r-integral by parts to find

$$\begin{aligned}
&\int\limits_{\phi(\epsilon)}^{\epsilon} r\log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{n,\epsilon}(r)\mathrm{d}r \\
&\qquad = \left(r\log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\right)_{r=\epsilon} - \left(r\log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\right)_{r=\phi(\epsilon)} \\
&\qquad\quad - \int\limits_{\phi(\epsilon)}^{\epsilon} \log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\mathrm{d}r - \int\limits_{\phi(\epsilon)}^{\epsilon} r\frac{\frac{\mathrm{d}}{\mathrm{d}r}\left|\varphi\left(re^{i\theta}\right)\right|}{\left|\varphi\left(re^{i\theta}\right)\right|} \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\mathrm{d}r,
\end{aligned}$$

and we remark that the derivative $\frac{\mathrm{d}}{\mathrm{d}r}\left|\varphi\left(re^{i\theta}\right)\right|$ is well-defined because the function $\varphi\left(re^{i\theta}\right)$ is assumed to be non-zero in the ball $B_a(0)$. Now, directly from the definition of $\rho_{n,\epsilon}(r)$ we have

$$\left(r\log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{n,\epsilon}(r)\right)_{r=\epsilon} = \epsilon\log\left|\varphi\left(\epsilon e^{i\theta}\right)\right| \left(\frac{\mathrm{d}}{\mathrm{d}r}\log\left(r\left(-\frac{\log(r)}{2\pi}\right)^{n}\right)\right)_{r=\epsilon}$$

$$
\begin{aligned}
&= \epsilon \log\left|\varphi\left(\epsilon e^{i\theta}\right)\right| \left(\frac{1}{\epsilon} + \frac{n}{\epsilon\log(\epsilon)}\right) \\
&= \log\left|\varphi\left(\epsilon e^{i\theta}\right)\right| + o(1) \qquad (\epsilon{\to}0)\,,
\end{aligned}
$$

and

$$
\left(r \log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r} \log \rho_{n,\epsilon}(r) \right)_{r=\phi(\epsilon)} = 0.
$$

Moreover, the third summand in the last expression contributes a term $o(1)$ to the asymptotic expansion for ϵ small by the estimate (4.2.11). The identical conclusion holds for the fourth summand, using the same argument with

$$
\max_{re^{i\theta}\in B_a(0)} \left| r \frac{\frac{\mathrm{d}}{\mathrm{d}r}\left|\varphi\left(re^{i\theta}\right)\right|}{\left|\varphi\left(re^{i\theta}\right)\right|} \right|
$$

in place of

$$
\max_{re^{i\theta}\in B_a(0)} \left| \log\left|\varphi\left(re^{i\theta}\right)\right| \right|.
$$

Therefore we have

$$
\int_0^{2\pi} \int_{\phi(\epsilon)}^{\epsilon} r \log\left|\varphi\left(re^{i\theta}\right)\right| \frac{\mathrm{d}^2}{\mathrm{d}r^2} \log\rho_{n,\epsilon}(r) \frac{\mathrm{d}r\wedge \mathrm{d}\theta}{2\pi} = \int_0^{2\pi} \log\left|\varphi\left(\epsilon e^{i\theta}\right)\right| \frac{\mathrm{d}\theta}{2\pi} + o(1) \qquad (\epsilon{\to}0)\,.
$$

Taking the limit for $\epsilon{\to}0$ of the remaining piace, using the continuity of $\varphi\left(re^{i\theta}\right)$, completes the proof of formula (4.2.11) and of the lemma. □

In the notation of proposition 1.3.15, the finite intersection numbers $\left(\overline{\mathcal{S}}_{k+1,\epsilon}, \overline{\mathcal{S}}_{k+1,\epsilon}\right)_{\mathrm{fin}}$, $\left(\overline{\mathcal{S}}_{k+1,\epsilon}, \overline{\omega}_{\mathcal{X},\epsilon}\right)_{\mathrm{fin}}$, and $(\overline{\omega}_{\mathcal{X},\epsilon}, \overline{\omega}_{\mathcal{X},\epsilon})_{\mathrm{fin}}$ are independent of ϵ. Therefore, the divergent behavior of the intersection numbers only comes from the contribution at infinity, and it is independent of the choice of sections. To shorten the presentation, we write

$$
\begin{aligned}
\left(\overline{S}_{k+1,\epsilon}, \overline{M}_{k,\epsilon}\right)_\infty &:= -\log\|l(z)\|_{\overline{S}_{k+1,\epsilon}}(\mathrm{div}(m)) - \int_X \log\|m(z)\|_{\overline{M}_{k,\epsilon}} \mathrm{c}_1\left(\overline{S}_{k+1,\epsilon}\right)(z) \\
&= \left(\overline{\mathcal{S}}_{k+1,\epsilon}, \overline{\mathcal{S}}_{k+1,\epsilon}\right)_\infty - \left(\overline{\mathcal{S}}_{k+1,\epsilon}, \overline{\omega}_{\mathcal{X},\epsilon}\right)_\infty.
\end{aligned}
$$

where l and m are the sections of S_{k+1} and M_k, respectively, induced by the complex parts of the sections implicitly chosen to define the contributions at infinity, and the equality is justified by the fact that ω_X is invertible and by the isometric isomorphism of hermitian line bundles

$$
\overline{M}_{k,\epsilon} \simeq \overline{S}_{k+1,\epsilon} \otimes \overline{\omega}_{X,\epsilon}^{\vee}.
$$

Similarly, we write

$$
\begin{aligned}
(\overline{\omega}_{X,\epsilon}, \overline{\omega}_{X,\epsilon})_{\infty} &:= -\log \|g(z)\|_{\mathrm{hyp},\epsilon} (\mathrm{div}(h)) - \int\limits_{X} \log \|h(z)\|_{\mathrm{hyp},\epsilon} \, \mathrm{c}_1 (\overline{\omega}_{X,\epsilon})(z) \\
&= (\overline{\omega}_{\mathcal{X},\epsilon}, \overline{\omega}_{\mathcal{X},\epsilon})_{\infty},
\end{aligned}
$$

where h and g are the complex parts of the sections chosen to define the corresponding arithmetic intersection products. We remark that, while the divergent terms of $\big(\overline{S}_{k+1,\epsilon}, \overline{M}_{k,\epsilon}\big)_{\infty}$ and $(\overline{\omega}_{X,\epsilon}, \overline{\omega}_{X,\epsilon})_{\infty}$ are independent of the sections chosen, this is not the case for the constant terms in their asymptotic expansions for ϵ small. In this notation, formula (4.2.1) is a corollary of the following proposition.

Proposition 4.2.4. *The asymptotic expansions*

$$
\big(\overline{S}_{k+1,\epsilon}, \overline{M}_{k,\epsilon}\big)_{\infty} = kp \, \log(-\log \epsilon) + O_{\Gamma,k}(1) \qquad (\epsilon \to 0)
$$

and

$$
(\overline{\omega}_{X,\epsilon}, \overline{\omega}_{X,\epsilon})_{\infty} = p \, \log(\epsilon) + 2p \, \log(-\log(\epsilon)) + O_{\Gamma}(1) \qquad (\epsilon \to 0)
$$

hold.

Proof. We begin by proving the first asymptotic expansion. Without loss of generality, we assume that there is only the cusp $i\infty$. Moreover, we assume that l and m are sections of S_{k+1} and M_k with divisors disjoint from each other and from the cusp. Then, in the ball $B_{s_\Gamma}(i\infty)$ with local coordinate q they have hyperbolic norms

$$
\|l(q)\|_{\overline{S}_{k+1}} = |q| \left(-\frac{\log |q|}{2\pi} \right)^{k+1} |\varphi_l(q)|,
$$

and

$$
\|m(q)\|_{\overline{M}_k} = \left(-\frac{\log |q|}{2\pi} \right)^{k} |\varphi_m(q)|,
$$

respectively, where $\varphi_l(q)$ and $\varphi_m(q)$ are the q-expansion of l divided by q and the q-expansion of m, respectively. Specifically, since we assumed the divisors of l and m to be disjoint from the cusp, the functions $\varphi_l(q)$ and $\varphi_m(q)$ are holomorphic on $B_{s_\Gamma}(i\infty)$ and non-zero at the cusp $i\infty$. The ϵ-regularized norms of the sections l and m in the ball $B_{s_\Gamma}(i\infty)$ are given by

$$
\|l(q)\|_{\overline{S}_{k+1},\epsilon} = \rho_{k+1,\epsilon}(|q|) \, |\varphi_l(q)|,
$$

and

$$
\|m(q)\|_{\overline{M}_k,\epsilon} = \frac{\rho_{k+1,\epsilon}(|q|)}{\rho_{1,\epsilon}(|q|)} \, |\varphi_m(q)|,
$$

respectively. We assumed $\operatorname{div}(m) \cap \{i\infty\} = \emptyset$, thus we have $\operatorname{div}(m) \cap B_\epsilon(i\infty) = \emptyset$ for ϵ small enough, and

$$-\log \|l(z)\|_{\overline{S}_{k+1,\epsilon}}(\operatorname{div}(m)) = -\log \|l(z)\|_{\overline{S}_{k+1}}(\operatorname{div}(m)) = O_{\Gamma,k}(1) \qquad (\epsilon \to 0)\,.$$

We remark that if this is not the case, i.e., if it does not hold $\operatorname{div}(m) \cap \{i\infty\} = \emptyset$, then the divergent term coming from the evaluation of $\log \|l(z)\|_{\overline{S}_{k+1,\epsilon}}$ at $i\infty$ cancels with an additional divergent term coming from the integral contribution. We now examine the asymptotics of the integral term. We restrict to ϵ small enough such that $\varphi_l(q)$ and $\varphi_m(q)$ do not have zeroes in $B_\epsilon(i\infty)$. Splitting the domain of integration and using the local coordinate q where appropriate, we compute

$$\begin{aligned}
&\big(\overline{S}_{k+1,\epsilon}, \overline{M}_{k,\epsilon}\big)_\infty \\
&\quad = -\int\limits_{X \backslash B_{s_\Gamma}(i\infty)} \log \|m(z)\|_{\overline{M}_k} \, \mathrm{c}_1(\overline{S}_{k+1})(z) - \int\limits_{B_{s_\Gamma}(i\infty) \backslash B_\epsilon(i\infty)} \log \|m(q)\|_{\overline{M}_k} \, \mathrm{c}_1(\overline{S}_{k+1})(q) \\
&\qquad - \int\limits_{B_\epsilon(i\infty) \backslash B_{\phi(\epsilon)}(i\infty)} \log \|m(q)\|_{\overline{M}_{k,\epsilon}} \, \mathrm{c}_1(\overline{S}_{k+1,\epsilon})(q) - \int\limits_{B_{\phi(\epsilon)}(i\infty)} \log \|m(q)\|_{\overline{M}_{k,\epsilon}} \, \mathrm{c}_1(\overline{S}_{k+1,\epsilon})(q) \\
&\qquad + O_{\Gamma,k}(1).
\end{aligned}$$

Let us examine the integrals occurring in the last expression. The first one is independent of ϵ and it contributes a term $O_{\Gamma,k}(1)$. In the second one the metrics do not depend on ϵ, and we use the expressions

$$\log \|m(q)\|_{\overline{M}_k} = \log((-\log |q|)^k) + \log \left(\frac{|\varphi_m(q)|}{(2\pi)^k} \right)$$

and

$$\mathrm{c}_1(\overline{S}_{k+1})(q) = \frac{k+1}{2\pi} \mu_{\mathrm{hyp}}(q).$$

In the third integral we use the expressions

$$\log \|m(q)\|_{\overline{M}_{k,\epsilon}} = \log \frac{\rho_{k+1,\epsilon}(|q|)}{\rho_{1,\epsilon}(|q|)} + \log(|\varphi_m(q)|)$$

and

$$\begin{aligned}
\mathrm{c}_1(\overline{S}_{k+1,\epsilon})(q) &= -\frac{i}{\pi} \partial_q \partial_{\bar q} \log \|l(q)\|_{\overline{S}_{k+1,\epsilon}} \mathrm{d}q \wedge \mathrm{d}\bar q \\
&= -\frac{i}{\pi} \partial_q \partial_{\bar q} \log \rho_{k+1,\epsilon}(|q|) \mathrm{d}q \wedge \mathrm{d}\bar q,
\end{aligned}$$

because $\partial_q \partial_{\bar q} \log(|\varphi_l(q)|) = 0$, since $\varphi_l(q)$ is holomorphic and non-zero. Finally, the last integral vanishes because, by construction, the ϵ-regularized Petersson metric is flat in the ball $B_{\phi(\epsilon)}(i\infty)$. Thus, we find

$$
\begin{aligned}
\left(\overline{S}_{k+1,\epsilon},\overline{M}_{k,\epsilon}\right)_{\infty}=&-\frac{k(k+1)}{2\pi}\int\limits_{B_{s_\Gamma}(i\infty)\setminus B_{\epsilon}(i\infty)}\log(-\log|q|)\,\frac{i}{2}\frac{\mathrm{d}q\wedge\mathrm{d}\bar{q}}{\left(|q|\log|q|\right)^{2}}\\
&-\frac{k+1}{2\pi}\int\limits_{B_{s_\Gamma}(i\infty)\setminus B_{\epsilon}(i\infty)}\log\left(\frac{|\varphi_m(q)|}{(2\pi)^k}\right)\frac{i}{2}\frac{\mathrm{d}q\wedge\mathrm{d}\bar{q}}{\left(|q|\log|q|\right)^{2}}\\
&+\frac{i}{\pi}\int\limits_{B_{\epsilon}(i\infty)\setminus B_{\phi(\epsilon)}(i\infty)}\log\frac{\rho_{k+1,\epsilon}(|q|)}{\rho_{1,\epsilon}(|q|)}\,\partial_q\partial_{\bar{q}}\log\rho_{k+1,\epsilon}(|q|)\mathrm{d}q\wedge\mathrm{d}\bar{q}\\
&+\frac{i}{\pi}\int\limits_{B_{\epsilon}(i\infty)\setminus B_{\phi(\epsilon)}(i\infty)}\log|\varphi_m(q)|\,\partial_q\partial_{\bar{q}}\log\rho_{k+1,\epsilon}(|q|)\mathrm{d}q\wedge\mathrm{d}\bar{q}+O_{\Gamma,k}(1).
\end{aligned}
$$

We pass to polar coordinates $q=re^{i\theta}$, observing that $|q|=r$, $\mathrm{d}q\wedge\mathrm{d}\bar{q}=-2ir\,\mathrm{d}r\wedge\mathrm{d}\theta$ and

$$
\partial_q\partial_{\bar{q}}\,f(|q|)=\frac{1}{4}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)f(r)\qquad\left(f\in\mathcal{C}^0(\mathbb{R}_{>0})\right).
$$

Therefore, we obtain

$$
\begin{aligned}
\left(\overline{S}_{k+1,\epsilon},\overline{M}_{k,\epsilon}\right)_{\infty}=&-\frac{k(k+1)}{2\pi}\int_0^{2\pi}\int_{\epsilon}^{s_\Gamma}\log(-\log(r))\,\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r\log(r)^2}\\
&-\frac{k+1}{2\pi}\int_0^{2\pi}\int_{\epsilon}^{s_\Gamma}\log\left(\frac{\left|\varphi_m\left(re^{i\theta}\right)\right|}{(2\pi)^k}\right)\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r\log(r)^2}\\
&+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon}\log\frac{\rho_{k+1,\epsilon}(r)}{\rho_{1,\epsilon}(r)}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\log\rho_{k+1,\epsilon}(r)\,\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}\\
&+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon}\log\left|\varphi_m\left(re^{i\theta}\right)\right|\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\log\rho_{k+1,\epsilon}(r)\,\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}+O_{\Gamma,k}(1).
\end{aligned}
$$

Since an antiderivative for the first integrand is given by the formula

$$
\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{1+\log(-\log(r))}{\log(r)}\right)=\frac{\log(-\log(r))}{r\log(r)^2},\tag{4.2.13}
$$

the first integral is convergent in the limit for $\epsilon\to 0$. Moreover, since $\varphi_m\left(re^{i\theta}\right)$ is holomorphic and non-zero at the cusp $i\infty$, also the second integral converges to a finite quantity in the limit for $\epsilon\to 0$. Performing the trivial θ-integrations, and decomposing the sum in the third and fourth integral, we obtain

$$
\begin{aligned}
\left(\overline{S}_{k+1,\epsilon}, \overline{M}_{k,\epsilon}\right)_\infty &= \int\limits_{\phi(\epsilon)}^{\epsilon} r \log \frac{\rho_{k+1,\epsilon}(r)}{\rho_{1,\epsilon}(r)} \frac{\mathrm{d}^2}{\mathrm{d}r^2} \log \rho_{k+1,\epsilon}(r) \mathrm{d}r \\
&+ \int\limits_{\phi(\epsilon)}^{\epsilon} \log \frac{\rho_{k+1,\epsilon}(r)}{\rho_{1,\epsilon}(r)} \frac{\mathrm{d}}{\mathrm{d}r} \log \rho_{k+1,\epsilon}(r) \mathrm{d}r \\
&+ \int\limits_{0}^{2\pi} \int\limits_{\phi(\epsilon)}^{\epsilon} r \log \left|\varphi_m\left(re^{i\theta}\right)\right| \frac{\mathrm{d}^2}{\mathrm{d}r^2} \log \rho_{k+1,\epsilon}(r) \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{2\pi} \\
&+ \int\limits_{0}^{2\pi} \int\limits_{\phi(\epsilon)}^{\epsilon} \log \left|\varphi_m\left(re^{i\theta}\right)\right| \frac{\mathrm{d}}{\mathrm{d}r} \log \rho_{k+1,\epsilon}(r) \frac{\mathrm{d}r \wedge \mathrm{d}\theta}{2\pi} + O_{\Gamma,k}(1) \\
&= k \log(-\log(\epsilon)) + O_{\Gamma,k}(1),
\end{aligned}
$$

where we used formulae (4.2.2), (4.2.3), (4.2.12) and (4.2.11), respectively.
We now prove the second asymptotic expansion. As in the previous case we assume, without loss of generality, that there is only the cusp $i\infty$. Moreover, we assume that g and h are sections of ω_X with divisors disjoint from each other and from the cusp, and that in the ball $B_{s_\Gamma}(i\infty)$ with local coordinate q they have hyperbolic norms

$$
\|g(q)\|_{\mathrm{hyp}} = |q|(-\log|q|)\,|\varphi_g(q)|,
$$

and

$$
\|h(q)\|_{\mathrm{hyp}} = |q|(-\log|q|)\,|\varphi_h(q)|,
$$

respectively, where $\varphi_g(q)$ and $\varphi_g(q)$ are holomorphic functions on $B_{s_\Gamma}(i\infty)$ that are non-zero at the cusp $i\infty$. Therefore, their ϵ-regularized norms in the ball B_{s_Γ} are

$$
\|g(q)\|_{\mathrm{hyp},\epsilon} = 2\pi\rho_{1,\epsilon}(|q|)\,|\varphi_g(q)|,
$$

and

$$
\|h(q)\|_{\mathrm{hyp},\epsilon} = 2\pi\rho_{1,\epsilon}(|q|)\,|\varphi_h(q)|,
$$

respectively. Since $\mathrm{div}(h) \cap \{i\infty\} \neq \emptyset$, we have $\mathrm{div}(h) \cap B_\epsilon(i\infty) = \emptyset$ for ϵ small enough, and

$$
-\log\|g(z)\|_{\mathrm{hyp},\epsilon}(\mathrm{div}(h)) = -\log\|g(z)\|_{\mathrm{hyp}}(\mathrm{div}(h)) = O_\Gamma(1).
$$

As in the previous case we remark that, if the condition $\mathrm{div}(h) \cap \{i\infty\} \neq \emptyset$ does not hold, then the divergent term coming from the evaluation of $\log\|g(z)\|_{\mathrm{hyp},\epsilon}$ at $i\infty$ cancels with an additional divergent term coming from the integral contribution. We restrict to ϵ small enough such that $\varphi_g(q)$ and $\varphi_h(q)$ do not have zeroes in $B_\epsilon(i\infty)$. Splitting the domain of integration, and using the local coordinate q where appropriate, we have

$$
\begin{aligned}
&(\overline{\omega}_{X,\epsilon},\overline{\omega}_{X,\epsilon})_\infty \\
&\quad= -\int\limits_{X\setminus B_{s_\Gamma}(i\infty)} \log\|h(z)\|_{\mathrm{hyp}}\,\mathrm{c}_1(\overline{\omega}_X)(z) - \int\limits_{B_{s_\Gamma}(i\infty)\setminus B_\epsilon(i\infty)} \log\|h(q)\|_{\mathrm{hyp}}\,\mathrm{c}_1(\overline{\omega}_X)(q) \\
&\qquad - \int\limits_{B_\epsilon(i\infty)\setminus B_{\phi(\epsilon)}(i\infty)} \log\|h(q)\|_{\mathrm{hyp},\epsilon}\,\mathrm{c}_1(\overline{\omega}_{X,\epsilon})(q) - \int\limits_{B_{\phi(\epsilon)}(i\infty)} \log\|h(q)\|_{\mathrm{hyp},\epsilon}\,\mathrm{c}_1(\overline{\omega}_{X,\epsilon})(q) \\
&\qquad + O_\Gamma(1).
\end{aligned}
$$

We examine the integrals occurring in the last expression. The first one is independent of ϵ and it is therefore $O_\Gamma(1)$. For the second one we use the relations

$$
\log\|h(q)\|_{\mathrm{hyp}} = \log(|q|(-\log|q|)) + \log|\varphi_h(q)|
$$

and $\mathrm{c}_1(\overline{\omega}_X)(q) = \frac{1}{2\pi}\mu_{\mathrm{hyp}}(q)$. For the third integral we have the expressions

$$
\log\|h(q)\|_{\mathrm{hyp},\epsilon} = \log\rho_{1,\epsilon}(|q|) + \log(2\pi\,|\varphi_h(q)|)
$$

and

$$
\mathrm{c}_1(\overline{\omega}_{X,\epsilon})(q) = -\frac{i}{\pi}\partial_q\partial_{\bar q}\log\rho_{1,\epsilon}(|q|)\,\mathrm{d}q\wedge\mathrm{d}\bar q.
$$

Finally, the fourth integral vanishes because the ϵ-regularized hyperbolic metric is flat on the ball $B_{\phi(\epsilon)}(i\infty)$ by construction. Therefore, we obtain

$$
\begin{aligned}
(\overline{\omega}_{X,\epsilon},\overline{\omega}_{X,\epsilon})_\infty = &-\frac{1}{2\pi}\int\limits_{B_{s_\Gamma}(i\infty)\setminus B_\epsilon(i\infty)} \log(|q|(-\log|q|))\,\frac{i}{2}\frac{\mathrm{d}q\wedge\mathrm{d}\bar q}{(|q|\log|q|)^2} \\
&-\frac{1}{2\pi}\int\limits_{B_{s_\Gamma}(i\infty)\setminus B_\epsilon(i\infty)} \log|\varphi_h(q)|\,\frac{i}{2}\frac{\mathrm{d}q\wedge\mathrm{d}\bar q}{(|q|\log|q|)^2} \\
&+\frac{i}{\pi}\int\limits_{B_\epsilon(i\infty)\setminus B_{\phi(\epsilon)}(i\infty)} \log\rho_{1,\epsilon}(|q|)\,\partial_q\partial_{\bar q}\log\rho_{1,\epsilon}(|q|)\,\mathrm{d}q\wedge\mathrm{d}\bar q \\
&+\frac{i}{\pi}\int\limits_{B_\epsilon(i\infty)\setminus B_{\phi(\epsilon)}(i\infty)} \log(2\pi\,|\varphi_h(q)|)\,\partial_q\partial_{\bar q}\log\rho_{1,\epsilon}(|q|)\,\mathrm{d}q\wedge\mathrm{d}\bar q + O_\Gamma(1).
\end{aligned}
$$

As in the previous computation, we pass to polar coordinates $q = re^{i\theta}$, and we find

$$
\begin{aligned}
(\overline{\omega}_{X,\epsilon},\overline{\omega}_{X,\epsilon})_\infty = &-\frac{1}{2\pi}\int_0^{2\pi}\int_\epsilon^{s_\Gamma}\log(-r\,\log(r))\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r\log(r)^2} - \frac{1}{2\pi}\int_0^{2\pi}\int_\epsilon^{s_\Gamma}\log\left|\varphi_h\big(re^{i\theta}\big)\right|\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{r\log(r)^2} \\
&+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon}\log\rho_{1,\epsilon}(r)\,\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\log\rho_{1,\epsilon}(r)\,\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}
\end{aligned}
$$

$$+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon} \log\left(2\pi\left|\varphi_h\left(re^{i\theta}\right)\right|\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\log\rho_{1,\epsilon}(r)\,\frac{r\,\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}+O_\Gamma(1).$$

We observe that the second integral in the last expression is convergent in the limit for $\epsilon\to 0$, because $\varphi_h\left(re^{i\theta}\right)$ is assumed to be non-zero at $i\infty$. Performing the trivial θ-integrations and rearranging terms, we reduce to the following contributions

$$\begin{aligned}(\overline{\omega}_{X,\epsilon},\overline{\omega}_{X,\epsilon})_\infty =& -\int_\epsilon^{s_\Gamma}\frac{\mathrm{d}r}{r\log(r)}-\int_\epsilon^{s_\Gamma}\log(-\log(r))\frac{\mathrm{d}r}{r\log(r)^2}\\ &+\int_{\phi(\epsilon)}^{\epsilon} r\,\log\rho_{1,\epsilon}(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{1,\epsilon}(r)\mathrm{d}r+\int_{\phi(\epsilon)}^{\epsilon}\log\rho_{1,\epsilon}(r)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{1,\epsilon}(r)\mathrm{d}r\\ &+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon} r\,\log\left(2\pi\left|\varphi_h\left(re^{i\theta}\right)\right|\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\log\rho_{1,\epsilon}(r)\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}\\ &+\int_0^{2\pi}\int_{\phi(\epsilon)}^{\epsilon}\log\left(2\pi\left|\varphi_h\left(re^{i\theta}\right)\right|\right)\frac{\mathrm{d}}{\mathrm{d}r}\log\rho_{1,\epsilon}(r)\frac{\mathrm{d}r\wedge\mathrm{d}\theta}{2\pi}+O_\Gamma(1).\end{aligned}$$

The first term can be explicitly integrated, and it yields the contribution

$$\log(-\log(\epsilon))+O_\Gamma(1).$$

The second term converges in the limit for $\epsilon\to 0$ by formula (4.2.13), and it contributes another factor $O_\Gamma(1)$. The asymptotic expansion of the last four terms is given by formulae (4.2.4), (4.2.5), (4.2.12) and (4.2.11) respectively. Summing up, we find

$$\begin{aligned}(\overline{\omega}_{X,\epsilon},\overline{\omega}_{X,\epsilon})_\infty &=\log(-\log(\epsilon))+\log(\epsilon)+\log(-\log(\epsilon))+O_\Gamma(1)\\ &=\log(\epsilon)+2\log(-\log(\epsilon))+O_\Gamma(1).\end{aligned}$$

We finally observe that if there is more than one cusp, each will produce precisely the same contribution to the asymptotic of the intersection numbers. This justifies the factor p in the statement of the proposition. □

Motivated by equation (4.2.1) we define a regularization of the intersection numbers.

Definition 4.2.5. We define the regularized intersection numbers by the limit

$$\begin{aligned}&\left(6\,\overline{\mathcal{S}}_{k+1}.\overline{\mathcal{S}}_{k+1}-6\overline{\mathcal{S}}_{k+1}.\overline{\omega}_{\mathcal{X}}+\overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right)^*\\ &\quad:=\lim_{\epsilon\to 0}\left(6\,\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\mathcal{S}}_{k+1,\epsilon}-6\overline{\mathcal{S}}_{k+1,\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}+\overline{\omega}_{\mathcal{X},\epsilon}.\overline{\omega}_{\mathcal{X},\epsilon}-p\log(\epsilon)-p\,\left(6k+2\right)\log(-\log(\epsilon))\right).\end{aligned}$$

Remark 4.2.6. Regularized arithmetic intersection numbers in this setting have been already defined by Hahn [31], who directly generalizes the work of Kühn [41]. Nevertheless equation (4.2.1) does not follow from the work of Hahn, because his regularization procedure involves cutting along a boundary instead of smoothening the metric.

Remark 4.2.7. As observed in remark 2.1.7, the line bundle of cusp forms $\overline{S}_{k+1}$ is not logarithmically singular. Therefore,

$$\widehat{\mathrm{c}}_1\left(\overline{\mathcal{S}}_{k+1}\right) \notin \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\log}),$$

where the latter group is discussed in [12, section 6]. A way to conceptualize the definition of regularized arithmetic intersection numbers of this section would be to find a complex $\mathcal{D}_{\text{fairly good}}$ such that

$$\widehat{\mathrm{c}}_1\left(\overline{\mathcal{S}}_{k+1}\right) \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\text{fairly good}}),$$

and show that the intersection numbers obtained using the general theory of [12] agree with the ad hoc definition discussed above.

4.3 Decomposition of the smoothened determinant

In this section we set up the strategy for studying the degeneration of the smoothened determinant $\det'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right)$. From the two previous sections we already know what is the divergent behavior of the smoothened determinant.

Corollary 4.3.1. *The asymptotic expansion*

$$\log\left(\det'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right)\right) = \frac{p}{6}\log(\epsilon) + \frac{p\,(3k+1)}{3}\log(-\log(\epsilon)) + O_{\Gamma,k}(1) \qquad (\epsilon\to 0)$$

holds.

Proof. The claimed asymptotic expansion follows by substituting into the smooth arithmetic Riemann–Roch theorem, formula (4.1.1), the asymptotic expansions of all the terms different than the smoothened determinant. Besides the terms independent of the metric, they are given in formulae (4.1.2), (4.1.3) and (4.2.1). □

Let us observe that remark 1.4.19 implies

$$\log\left(\det'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right)\right) = \log\left(\det'\left(\Delta^0_{\overline{M}_{-k,\epsilon}}\right)\right).$$

Remark 4.3.2. The first statement of proposition 4.19 of [26] establishes the asymptotic expansion of the determinant of the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}}$ on functions through a metric degeneration comparable to the one defined in this work. Since, see equation (4.2) of loc. cit., the determinant of this Laplacian is related to the determinant of the Laplace–Beltrami operator Δ_0 via the formula

$$\det'(\Delta_{\bar{\partial}}) = 2^{\frac{g+2}{3}}\det'(\Delta_0),$$

one would expect the given expansion to match the one of corollary 4.3.1 when they both apply. Indeed, setting $k=0$ in the corollary above and excluding the presence of elliptic points in the statement of loc. cit., i.e., taking $\sum_j\left(1-m_j^{-1}\right)^2 = c = p$ in their formula, the two asymptotic expansions agree.

Definition 4.3.3. In analogy with the assignment given in (2.2.1), we define

$$\Delta_{k,\epsilon} := 4\Delta^0_{\overline{M}_{-k,\epsilon}},$$

and we denote by $K^\Gamma_{k,\epsilon}(t;z,w)$ the associated heat kernel on X.

Proposition 4.3.4. *We have the relation of regularized determinants*

$$\log\left(\det{}'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right)\right) = \log\left(\det{}'\left(\Delta_{k,\epsilon}\right)\right) + \left(\frac{(2g-2)(3k+1)}{3} + p\,k + 2N_k\right)\log(2).$$

Proof. By remark 1.4.19 and observation 1.4.17 we have

$$\det{}'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right) = \det{}'\left(\Delta^0_{\overline{M}_{-k,\epsilon}}\right) = 4^{-\zeta^\Gamma_{k,\epsilon}(0)}\det{}'\left(\Delta_{k,\epsilon}\right),$$

where $\zeta^\Gamma_{k,\epsilon}(s)$ is the spectral ζ-function associated to the Laplacian $\Delta_{k,\epsilon}$. Therefore the statement is equivalent to the evaluation

$$\zeta^\Gamma_{k,\epsilon}(0) = -\frac{(2g-2)(3k+1)}{6} - \frac{p\,k}{2} - N_k. \tag{4.3.1}$$

Fay gives in [22, theorem 2.5] the first two coefficients of the asymptotic expansion for $t\to 0$ of the heat kernel associated to a smooth hermitian holomorphic vector bundle. Integrating this expansion on a compact Riemann surface, as it is done on page 37 of loc. cit., one obtains the asymptotic expansion

$$\int\limits_X K^\Gamma_{k,\epsilon}(t;z,z)\mu_\epsilon(z) = \frac{\mathrm{vol}_\epsilon(X)}{4\pi t} + \frac{1}{3}(1-g) + \frac{1}{2}\deg\left(M_{-k}\right) + O_{\Gamma,k,\epsilon}(t) \qquad (t\to 0)\,.$$

The degree of M_{-k} is computed using the isomorphism of observation 2.1.6, together with $\deg(\omega_X) = 2g-2$ and $\deg(\mathcal{O}_X(D)) = p$, yielding

$$\deg\left(M_{-k}\right) = \deg\left(\left(\omega_X\otimes\mathcal{O}_X(D)\right)^{\otimes -k}\right) = -k\left(\deg(\omega_X) + \deg(\mathcal{O}_X(D))\right) = -k(2g-2+p).$$

Therefore

$$\int\limits_X K^\Gamma_{k,\epsilon}(t;z,z)\mu_\epsilon(z) = \frac{\mathrm{vol}_\epsilon(X)}{4\pi t} - \frac{(2g-2)(3k+1)}{6} - \frac{p\,k}{2} + O_{\Gamma,k,\epsilon}(t) \qquad (t\to 0)\,.$$

Regarding the large time asymptotics for the trace of the heat kernel $K^\Gamma_{k,\epsilon}(t;z,z)$ we first observe that, as stated on page 24 of [22], the spectrum of $\Delta_{k,\epsilon}$ is non-negative. Therefore, using the isomorphism

$$\ker(\Delta^0_{\overline{M}_{-k,\epsilon}}) \simeq \mathrm{H}^0(X, M_{-k})$$

given by Hodge theory, and the evaluation of N_k given in lemma 2.6.3, we find

$$\begin{aligned} N_{k,\epsilon} &:= \lim_{t\to\infty}\int\limits_X K^\Gamma_{k,\epsilon}(t;z,z)\mu_\epsilon(z) = \dim\ker(\Delta_{k,\epsilon}) = \dim\ker\left(\Delta^0_{\overline{M}_{-k,\epsilon}}\right) \\ &= \dim\mathrm{H}^0\left(X, M_{-k}\right) = \left\{\begin{matrix} 1, & k=0 \\ 0, & k\geq 1 \end{matrix}\right\} = N_k. \end{aligned}$$

Thus, we obtain the integral representation

$$\zeta_{k,\epsilon}^{\Gamma}(s) = \frac{1}{\Gamma(s)}\mathcal{M}\left(\operatorname{Tr} K_{k,\epsilon}^{\Gamma}(t), s\right) = \int\limits_{0}^{\infty}\left(\operatorname{Tr} K_{k,\epsilon}^{\Gamma}(t) - N_k\right) t^{s-1}\mathrm{d}t \qquad (\operatorname{Re}(s) > 1)\,.$$

Applying the direct mapping theorem for the Mellin transform, theorem B.2, we find the Laurent expansion

$$\mathcal{M}\left(\operatorname{Tr} K_{k,\epsilon}^{\Gamma}(t), s\right) = \frac{1}{s}\left(-\frac{(2g-2)(3k+1)}{6} - \frac{p\,k}{2} - N_k\right) + O_{\Gamma,k,\epsilon}(1) \qquad (s\to 0)\,,$$

from which the special value of the spectral ζ-function follows. □

Remark 4.3.5. By observation 2.6.6 and equation (4.3.1), we have

$$\zeta_k^{\Gamma}(0) - \zeta_{k,\epsilon}^{\Gamma}(0) = -\frac{p}{6}.$$

Now, using the linearity of the Mellin transform, we write the decomposition

$$\begin{aligned}
-\log&\left(\det{}'\left(\Delta_{k,\epsilon}\right)\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)} K_k^{\Gamma}(t;z,z)\,\mu_\epsilon(z), s\right)\right)\Bigg|_{s=0} && \text{(A)}\\
&\quad+ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)} \left(K_{k,\epsilon}^{\Gamma}(t;z,z) - K_k^{\Gamma}(t;z,z)\right)\mu_\epsilon(z), s\right)\right)\Bigg|_{s=0} && \text{(B)}\\
&\quad+ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)} K_{k,\epsilon}^{\Gamma}(t;z,z)\,\mu_\epsilon(z), s\right)\right)\Bigg|_{s=0}. && \text{(C)}
\end{aligned}$$

We further decompose the term (A). On its domain of integration holds the equality $\mu_\epsilon(z) = \mu_{\mathrm{hyp}}(z)$. By formulae (2.2.4) and (2.4.1), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)} K_k^{\Gamma}(t;z,z)\,\mu_\epsilon(z), s\right)\right)\Bigg|_{s=0}\\
&\quad= \frac{\mathrm{d}}{\mathrm{d}s}\Bigg(\frac{1}{\Gamma(s)}\mathcal{M}\Bigg(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)} \sum_{\gamma\in H(\Gamma)\cup P(\Gamma)}\ \sum_{[\eta]\in\Gamma_\gamma\setminus\Gamma}\ \sum_{n=1}^{\infty} j(\eta^{-1}\gamma^n\eta, z)^k\left(\frac{\eta^{-1}\gamma^n\eta(z) - \bar{z}}{z - \eta^{-1}\gamma^n\eta(\bar{z})}\right)^k\\
&\qquad\qquad \times K_k(t; d_{\mathrm{hyp}}(z, \eta^{-1}\gamma^n\eta(z))) + K_k(t;0)\,\mu_{\mathrm{hyp}}(z), s\Bigg)\Bigg)\Bigg|_{s=0}.
\end{aligned}$$

Unfolding the integration domain for the parabolic elements, and conjugating it by the scaling matrix of the respective cusp, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}K_k^{\Gamma}(t;z,z)\,\mu_\epsilon(z),s\right)\right)_{s=0}$$

$$=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(\mathrm{H}K_k^{\Gamma}(t;z,z)+K_k(t;0)\right)\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0} \tag{AHyp}$$

$$+\,p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\setminus\bigcup\limits_{[\eta]\in\Gamma_\infty\setminus\Gamma}\bigcup_{j=1}^{p}\eta(B_\epsilon(P_j))}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k \right.\right. \tag{APar}$$

$$\left.\left.\times\,K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0}.$$

Without loss of generality, we assume $P_1=i\infty$. Then we shrink ϵ, if needed, to ensure

$$B_\infty(\epsilon):=\bigcup_{\substack{[\eta]\in\Gamma_\infty\setminus\Gamma\\ [\eta]\neq\Gamma_\infty}}\left(\bigcup_{j=1}^{p}\eta\left(B_\epsilon(P_j)\right)\right)\cup\bigcup_{j=2}^{\infty}B_\epsilon(P_j)\subset\mathcal{F}_\infty(y<1),$$

for which we obtain the equality

$$\mathcal{F}_\infty\setminus\bigcup_{[\eta]\in\Gamma_\infty\setminus\Gamma}\bigcup_{j=1}^{p}\eta\left(B_\epsilon(P_j)\right)=\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)\setminus B_\infty(\epsilon).$$

Then, splitting the domain of integration at $y=1$, and adding and subtracting the value $n=0$ in the sum, we further decompose the term (APar) as

$$p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0}$$

$$=p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)}K_k^{\infty}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0} \tag{APar1}$$

$$-\,p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)}K_k(t;0)\,\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0} \tag{APar2}$$

$$+\,p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k \right.\right. \tag{APar3}$$

$$\left.\left.\times\,K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0}.$$

Summing up, and, by abuse of notation, denoting each term by the corresponding equation, we have the decomposition of the Quillen correction term

$$-\log\left(\det{}'\left(\Delta_{k,\epsilon}\right)\right)=(\mathrm{APar1})+(\mathrm{APar2})+(\mathrm{APar3})+(\mathrm{AHyp})+(\mathrm{B})+(\mathrm{C}). \tag{4.3.2}$$

In view of this decomposition, we refer to the sum of the terms (APar1), (APar2) and (APar3) as the parabolic contribution to the smoothened determinant, to (AHyp) as the hyperbolic contribution to the smoothened determinant, and to the terms (B) and (C) as the rest terms.

4.4 Degeneration of the parabolic contribution to the smoothened determinant

In this section we provide an asymptotic expansion for the parabolic contribution (APar) to the smoothened determinant, which is partially implicit. It accounts for the k-dependent terms in the diverging part of the asymptotic expansion of the smoothened determinant given in corollary 4.3.1. To avoid excessively lengthy expressions we use the notation

$$\begin{aligned}\mathrm{Rest}_{\mathrm{par}}(\epsilon) :=& \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{4\pi^3 n}\int_0^{\infty} r\sinh(2\pi r)e^{-tr^2}\right.\right.\\ &\left.\left.\times\sum_{\kappa=\pm k}\left|\Gamma\left(\kappa+\frac{1}{2}+ir\right)\right|^2 W_{-\kappa,ir}(4\pi ny)^2\,\mathrm{d}r\frac{\mathrm{d}y}{y^2},s\right)\right)\Bigg|_{s=0}\end{aligned}$$

for the term whose asymptotic expansion will not be given. Now, we use theorem 3.4.1 to provide an asymptotic expansion for (APar1).

Proposition 4.4.1. *The term* (APar1) *has the asymptotic expansion*

$$\begin{aligned}p\frac{\mathrm{d}}{\mathrm{d}s}&\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)}K_k^{\infty}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}\\ &= p\,\mathrm{Rest}_{\mathrm{par}}(\epsilon) - p\left(k+\frac{1}{2}\right)\log(-\log(\epsilon)) + p\,O_k(1) \qquad (\epsilon\to 0),\end{aligned}$$

where the implied constant term has the value

$$p\left(k+\frac{1}{2}\right)\log(2\pi) - p\sum_{j=0}^{k-1}\frac{(2k-2j-1)\log((2k-j)(j+1))}{\Gamma(2k-j)\,\Gamma(j+1)}\sum_{n=1}^{\infty}\int_{4\pi n}^{\infty}W_{k,k-j-\frac{1}{2}}(v)^2\frac{\mathrm{d}v}{v^2}.$$

Proof. By theorem 3.4.1, performing the trivial x-integration we have the decomposition

$$\begin{aligned}p\frac{\mathrm{d}}{\mathrm{d}s}&\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)}K_k^{\infty}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}\\ &= p\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}}\frac{\mathrm{d}y}{y},s\right)\right)\Bigg|_{s=0} \qquad (4.4.1)\end{aligned}$$

$$+\,p\,\frac{\mathrm{d}}{\mathrm{d}s}\Bigg(\frac{1}{\Gamma(s)}\mathcal{M}\Bigg(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{4\pi n}\sum_{j=0}^{k-1}\frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\,\Gamma(j+1)} \tag{4.4.2}$$

$$\times\, W_{k,k-j-\frac{1}{2}}(4\pi n y)^2\frac{\mathrm{d}y}{y^2},s\Bigg)\Bigg)\Bigg|_{s=0} + p\,\mathrm{Rest}_{\mathrm{par}}(\epsilon).$$

Regarding the first term (4.4.1), we observe that the variables t and y are separated. Thus we compute

$$\begin{aligned}
&p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}}\frac{\mathrm{d}y}{y},s\right)\right)\Bigg|_{s=0}\\
&\qquad\qquad= \frac{p}{\sqrt{4\pi}}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\frac{\mathrm{d}y}{y}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t\left(k+\frac{1}{2}\right)^2},s-\frac{1}{2}\right)\right)\Bigg|_{s=0}\\
&\qquad\qquad= \frac{p}{\sqrt{4\pi}}\left(\log(-\log(\epsilon))-\log(2\pi)\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)\Bigg|_{s=0}.
\end{aligned}$$

Using the functional equation of the Γ-function, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)\Bigg|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}s\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\right)\Bigg|_{s=0}\\
&= \left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\right)\Bigg|_{s=0}\\
&\quad+\left(\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\right)s\right)\Bigg|_{s=0},
\end{aligned}$$

and we observe that the second term in the last expression is the derivative of a holomorphic function times s, all evaluated at $s=0$, therefore it vanishes. We deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\left(k+\frac{1}{2}\right)^{1-2s}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)\Bigg|_{s=0} = \frac{\left(k+\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\Gamma(1)} = -2\sqrt{\pi}\left(k+\frac{1}{2}\right).$$

Using this computation we obtain

$$p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{\sqrt{4\pi t}}\frac{\mathrm{d}y}{y},s\right)\right)\Bigg|_{s=0} = -p\left(k+\frac{1}{2}\right)\left(\log(-\log(\epsilon))-\log(2\pi)\right).$$

We now compute the contribution coming from the term (4.4.2). Also in this case the variables t and y are separated. Moreover, since all the terms involved in the integration are strictly positive, we use Tonelli's theorem to exchange integrals. Thus, we have

$$
\begin{aligned}
&p\,\frac{\mathrm{d}}{\mathrm{d}s}\Bigg(\frac{1}{\Gamma(s)}\mathcal{M}\Bigg(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{4\pi n}\sum_{j=0}^{k-1}\frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\,\Gamma(j+1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times W_{k,k-j-\frac{1}{2}}(4\pi ny)^2\frac{\mathrm{d}y}{y^2},s\Bigg)\Bigg)\Bigg|_{s=0} \\
&= p\sum_{j=0}^{k-1}\frac{(2k-2j-1)}{\Gamma(2k-j)\,\Gamma(j+1)}\left(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{4\pi n}W_{k,k-j-\frac{1}{2}}(4\pi ny)^2\frac{\mathrm{d}y}{y^2}\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t(2k-j)(j+1)},s\right)\right)\Bigg|_{s=0}.
\end{aligned}
$$

Rewriting the Mellin transform as a Γ-function and simplifying, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(e^{-t(2k-j)(j+1)},s\right)\right)\Bigg|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(((2k-j)(j+1))^{-s}\right)_{s=0} \\
&= -\log((2k-j)(j+1)).
\end{aligned}
$$

On the other hand, using the change of variable $v=4\pi ny$ and the asymptotic expansion (3.4.1), we find

$$
\lim_{\epsilon\to 0}\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{4\pi n}W_{k,k-j-\frac{1}{2}}(4\pi ny)^2\frac{\mathrm{d}y}{y^2}=\sum_{n=1}^{\infty}\int_{4\pi n}^{\infty}W_{k,k-j-\frac{1}{2}}(v)^2\frac{\mathrm{d}v}{v^2}<\infty.
$$

Summing up, we have the asymptotic expansion

$$
\begin{aligned}
&p\,\frac{\mathrm{d}}{\mathrm{d}s}\Bigg(\frac{1}{\Gamma(s)}\mathcal{M}\Bigg(\int_1^{-\frac{\log(\epsilon)}{2\pi}}\sum_{n=1}^{\infty}\frac{1}{4\pi n}\sum_{j=0}^{k-1}\frac{(2k-2j-1)e^{-t(2k-j)(j+1)}}{\Gamma(2k-j)\,\Gamma(j+1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times W_{k,k-j-\frac{1}{2}}(4\pi ny)^2\frac{\mathrm{d}y}{y^2},s\Bigg)\Bigg)\Bigg|_{s=0} \\
&= -p\sum_{j=0}^{k-1}\frac{(2k-2j-1)\log((2k-j)(j+1))}{\Gamma(2k-j)\,\Gamma(j+1)}\sum_{n=1}^{\infty}\int_{4\pi n}^{\infty}W_{k,k-j-\frac{1}{2}}(v)^2\frac{\mathrm{d}v}{v^2}+o(1)\qquad(\epsilon\to 0).
\end{aligned}
$$

The asymptotic behavior of the remaining term $p\,\mathrm{Rest}_{\mathrm{par}}(\epsilon)$ is left implicit. □

Observation 4.4.2. Given the expressions appearing in the term $\mathrm{Rest}_{\mathrm{par}}(\epsilon)$ and in the last proposition, and that are left implicit, it is possible that the following integration formula can be used to explicitly evalute these terms. We have

$$
\int W_{\kappa,\mu}(Z)^2\frac{\mathrm{d}Z}{Z^2}=\frac{1}{\mu\,Z}\left(\frac{\mathrm{d}}{\mathrm{d}\mu}W_{\kappa,\mu}(Z)\,W_{\kappa+1,\mu}(Z)-W_{\kappa,\mu}(Z)\,\frac{\mathrm{d}}{\mathrm{d}\mu}W_{\kappa+1,\mu}(Z)\right).
$$

Proof. Formula (4γ) on page 113 of [10], specialized to the function $W_{\kappa,\mu}(Z)$, states

$$\int W_{\kappa,\mu}(Z)^2 \frac{\mathrm{d}Z}{Z^2} = \frac{1}{\mu}\left(W_{\kappa,\mu}(Z)\,\frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}}{\mathrm{d}Z}W_{\kappa,\mu}(Z) - \frac{\mathrm{d}}{\mathrm{d}\mu}W_{\kappa,\mu}(Z)\,\frac{\mathrm{d}}{\mathrm{d}Z}W_{\kappa,\mu}(Z)\right).$$

Moreover, the derivation formula 13.5.26 of [48], namely

$$\left(Z\frac{\mathrm{d}}{\mathrm{d}Z}Z\right)^n\left(e^{-\frac{Z}{2}}Z^{\kappa-1}W_{\kappa,\mu}(Z)\right) = (-1)^n e^{\frac{Z}{2}}Z^{\kappa+n-1}W_{\kappa,\mu}(Z) \qquad (n\in\mathbb{N}), \tag{4.4.3}$$

implies

$$\frac{\mathrm{d}}{\mathrm{d}Z}W_{\kappa,\mu}(Z) = \left(\frac{1}{2}-\frac{\kappa}{Z}\right)W_{\kappa,\mu}(Z) - \frac{1}{Z}W_{\kappa+1,\mu}(Z).$$

The observation follows by simplifying the expression. □

We now examine the term (APar2).

Proposition 4.4.3. *The following asymptotic expansion holds*

$$-p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} K_k(t;0)\,\mu_{\mathrm{hyp}}(z), s\right)\right)\Bigg|_{s=0} = -p\,c_k + o(1) \qquad (\epsilon\to 0).$$

Here c_k are the constants defined in formula (2.8.7) *and computed in proposition* 2.8.3.

Proof. Since $K_k(t;0)$ is independent of z, we have

$$\begin{aligned}-p\,\frac{\mathrm{d}}{\mathrm{d}s}&\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} K_k(t;0)\,\mu_{\mathrm{hyp}}(z), s\right)\right)\Bigg|_{s=0}\\ &= -p\,\mathrm{vol}_{\mathrm{hyp}}\left(\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(K_k(t;0),s\right)\right)\Bigg|_{s=0}.\end{aligned}$$

By formula (2.8.7) the last factor equals c_k. And, by explicit computation, we have

$$\mathrm{vol}_{\mathrm{hyp}}\left(\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)\right) = 1 + \frac{2\pi}{\log(\epsilon)}.$$

Taking the limit for $\epsilon\to 0$ completes the proof of the proposition. □

It only remains to study the degeneration of the term (APar3). As in the proof of proposition 2.5.4, we use the notation

$$d(z,n) := d_{\mathrm{hyp}}(z,z+n) = \operatorname{arccosh}\left(1+\frac{n^2}{2y^2}\right)$$

for the next technical lemmata. Moreover, we observe the elementary bounds

$$\log\left(\frac{n^2}{y^2}+1\right) \le d(z,n) \le 2\log\left(\frac{n^2}{y^2}+1\right). \tag{4.4.4}$$

Lemma 4.4.4. *The estimate*

$$\int\limits_0^1 \sum_{n=1}^{\infty} d(z,n) e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll t^{\frac{3}{2}} e^{-\frac{t}{4}}$$

holds.

Proof. The proof is a straightforward calculation. First we rearrange terms and we apply the bounds (4.4.4), the upper one for the linear term in $d(z,n)$ and the lower one for the exponential term in $d(z,n)$, to obtain

$$\int\limits_0^1 \sum_{n=1}^{\infty} d(z,n) e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll \int\limits_0^1 \sum_{n=1}^{\infty} \log\left(\frac{n^2}{y^2}+1\right) e^{-\frac{\left(t+\log\left(\frac{n^2}{y^2}+1\right)\right)^2}{4t}} \frac{\mathrm{d}y}{y^2}.$$

We now expand the form of the exponential. Then we use $n^2+y^2 \approx (n+1)^2$, and, by non-negativity of the terms to be added and integrated, we exchange series and integral to obtain

$$\begin{aligned}\int\limits_0^1 \sum_{n=1}^{\infty} d(z,n) e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} &\ll e^{-\frac{t}{4}} \int\limits_0^1 \sum_{n=1}^{\infty} \log\left(\frac{n^2}{y^2}+1\right) \left(\frac{y^2}{n^2+y^2}\right)^{\frac{1}{2}} e^{-\frac{\log\left(\frac{n^2}{y^2}+1\right)^2}{4t}} \frac{\mathrm{d}y}{y^2} \\ &\ll e^{-\frac{t}{4}} \sum_{n=1}^{\infty} \frac{1}{n+1} \int\limits_0^1 \log\left(\frac{n^2}{y^2}+1\right) e^{-\frac{\log\left(\frac{n^2}{y^2}+1\right)^2}{4t}} \frac{\mathrm{d}y}{y}.\end{aligned}$$

Changing variable to $v = \frac{\log\left(\frac{n^2}{y^2}+1\right)}{2\sqrt{t}}$, and using the estimate

$$\frac{\mathrm{d}y}{y} = -\frac{\mathrm{d}\left(\frac{n^2}{y^2}+1\right)}{\frac{2n^2}{y^2}} \approx -\frac{\mathrm{d}\left(\frac{n^2}{y^2}+1\right)}{\frac{n^2}{y^2}+1} = -2\sqrt{t}\,\mathrm{d}v, \tag{4.4.5}$$

we obtain

$$\int\limits_0^1 \sum_{n=1}^{\infty} d(z,n) e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll t\, e^{-\frac{t}{4}} \sum_{n=1}^{\infty} \frac{1}{n+1} \int\limits_{\frac{\log(n^2+1)}{2\sqrt{t}}}^{\infty} v\, e^{-v^2} \mathrm{d}v.$$

The integral can be now explicitly calculated, the result is

$$\int\limits_{\frac{\log(n^2+1)}{2\sqrt{t}}}^{\infty} v\, e^{-v^2} \mathrm{d}v = \frac{1}{2} e^{-\frac{\log\left(n^2+1\right)^2}{4t}} < e^{-\frac{\log(n+1)^2}{4t}}.$$

Since the function $\frac{1}{u}e^{-\frac{\log(u)^2}{4t}}$ is strictly monotone decreasing, we can bound the sum in n by the corresponding integral and find

$$\int_0^1 \sum_{n=1}^{\infty} d(z,n)e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll t\,e^{-\frac{t}{4}} \sum_{n=1}^{\infty} \frac{1}{n+1} e^{-\frac{\log(n+1)^2}{4t}} < t\,e^{-\frac{t}{4}} \int_1^{\infty} e^{-\frac{\log(n)^2}{4t}} \frac{\mathrm{d}n}{n}.$$

The change of variable $u = \frac{\log(n)}{2\sqrt{t}}$ completes the calculation. □

We now consider the quantity

$$\mathrm{I}_k(t) := \int_0^1 \sum_{n=1}^{\infty} K_k(t; d(z,n)) \frac{\mathrm{d}y}{y^2}.$$

Lemma 4.4.5. *For each $k \geq 0$ and $t > 0$ the function $\mathrm{I}_k(t)$ is well-defined and continuous. Moreover, for $t \geq 1$ if $k = 0$ and $t \geq \frac{\log(2)}{2k-1}$ if $k \geq 1$, we have*

$$\mathrm{I}_k(t) \ll_k e^{-\frac{t}{5}}. \tag{4.4.6}$$

And, for $0 < t \leq 1$ if $k = 0$ and $0 < t \leq \frac{\log(2)}{2k-1}$ if $k \geq 1$, we have

$$\mathrm{I}_k(t) \ll_k e^{-\frac{1}{20\,t}}. \tag{4.4.7}$$

Proof. Applying lemma 2.3.2, we have the decomposition

$$\mathrm{I}_k(t) \approx_k \int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \tag{4.4.8}$$

$$+ \int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2}. \tag{4.4.9}$$

We provide separate estimates for six different cases. Namely, we estimate the term (4.4.8) separating the case t small from the case t large, and we estimate the term (4.4.9) separating the case t small from the case t large, and further separating the cases $k = 0$ and $k \geq 1$.

Estimate of (4.4.8) *for t large.* Applying the estimate (2.3.3) and lemma 4.4.4 we compute

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll \int_0^1 \sum_{n=1}^{\infty} \frac{d(z,n)\,e^{-tk(k+1)-\frac{(t+d(z,n))^2}{4t}}}{t^{\frac{3}{2}}} \frac{\mathrm{d}y}{y^2}$$

$$\ll \frac{e^{-tk(k+1)}}{t^{\frac{3}{2}}} \int_0^1 \sum_{n=1}^{\infty} d(z,n)\,e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll e^{-t\left(k+\frac{1}{2}\right)^2}.$$

Estimate of (4.4.9) *for t large and $k = 0$.* We apply the bound (2.3.4) and find

$$\int_0^1 \sum_{n=1}^{\infty} A_0^{(1,\infty)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} \ll \int_0^1 \sum_{n=1}^{\infty} \frac{e^{-\frac{(t+d(z,n)+1)^2}{4t}}}{\sqrt{t}} \frac{\mathrm{d}y}{y^2}.$$

We observe $e^{-\frac{(t+d(z,n)+1)^2}{4t}} < e^{-\frac{(t+d(z,n))^2}{4t}}$ and $d(z,n) > \log(2) \gg 1$, then we apply lemma 4.4.4. We have

$$\int_0^1 \sum_{n=1}^{\infty} A_0^{(1,\infty)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} \ll \frac{1}{\sqrt{t}} \int_0^1 \sum_{n=1}^{\infty} d(z,n) e^{-\frac{(t+d(z,n))^2}{4t}} \frac{\mathrm{d}y}{y^2} \ll\, t e^{-\frac{t}{4}} \ll e^{-\frac{t}{5}}.$$

Estimate of (4.4.9) *for t large and $k \geq 1$.* We apply the bound (2.3.5) and obtain

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} \ll_k \int_0^1 \sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{t}}\right) e^{-(2t+d(z,n))k} \frac{\mathrm{d}y}{y^2},$$

Since we assumed $1 \ll_k t$, it holds $\left(1 + \frac{1}{\sqrt{t}}\right) \ll_k 1$. Further applying the lower bound of formula (4.4.4), we find

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} \ll_k e^{-2tk} \int_0^1 \sum_{n=1}^{\infty} \left(\frac{y^2}{n^2 + y^2}\right)^k \frac{\mathrm{d}y}{y^2} \ll_k\, e^{-2tk}.$$

This completes the proof the estimate for t large (4.4.6).

Estimate of (4.4.8) *for t small.* Recall that we assumed $0 < t \leq \frac{\log(2)}{2k-1}$ for $k \geq 1$ and $0 < t \leq 1$ for $k = 0$. Using the expression (2.3.1) we find

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} = \int_0^1 \sum_{n=1}^{\infty} \frac{d(z,n)\, e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{d(z,n)}{2}}}{t^{\frac{3}{2}}} \int_0^1 \frac{e^{-\frac{(v+d(z,n))^2}{4t}}}{\sqrt{v}} \mathrm{d}v\, \frac{\mathrm{d}y}{y^2}.$$

By Tonelli's theorem, we exchange the order of integrals and series, and we rearrange terms. We thus have

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n))\, \frac{\mathrm{d}y}{y^2} = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{t^{\frac{3}{2}}} \int_0^1 \frac{1}{\sqrt{v}} \sum_{n=1}^{\infty} \int_0^1 d(z,n)\, e^{-\frac{(v+d(z,n))^2}{4t} - \frac{d(z,n)}{2}} \frac{\mathrm{d}y}{y^2}\, \mathrm{d}v.$$

Now we apply the bounds (4.4.4) for $d(z,n)$, the lower one for the exponential term in $d(z,n)$ and the upper one for the linear term $d(z,n)$, and we observe $e^{-t\left(k+\frac{1}{2}\right)^2} \ll 1$ and $n^2 + y^2 \approx (n+1)^2$. We obtain

$$
\begin{aligned}
&\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \, \frac{\mathrm{d}y}{y^2} \\
&\qquad = \frac{e^{-t\left(k+\frac{1}{2}\right)^2}}{t^{\frac{3}{2}}} \int_0^1 \frac{1}{\sqrt{v}} \sum_{n=1}^{\infty} \int_0^1 d(z,n) \, e^{-\frac{v^2}{4t} - \frac{d(z,n)^2}{4t} - d(z,n)\left(\frac{v}{2t}+\frac{1}{2}\right)} \frac{\mathrm{d}y}{y^2} \, \mathrm{d}v \\
&\qquad \ll \frac{1}{t^{\frac{3}{2}}} \int_0^1 \frac{e^{-\frac{v^2}{4t}}}{\sqrt{v}} \sum_{n=1}^{\infty} \int_0^1 \log\left(\frac{n^2}{y^2}+1\right) \left(\frac{y^2}{n^2+y^2}\right)^{\frac{v}{2t}+\frac{1}{2}} e^{-\frac{\log\left(\frac{n^2}{y^2}+1\right)^2}{4t}} \frac{\mathrm{d}y}{y^2} \, \mathrm{d}v \\
&\qquad \ll \frac{1}{t^{\frac{3}{2}}} \int_0^1 \frac{e^{-\frac{v^2}{4t}}}{\sqrt{v}} \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{v}{t}+1}} \int_0^1 \log\left(\frac{n^2}{y^2}+1\right) y^{\frac{v}{t}} \, e^{-\frac{\log\left(\frac{n^2}{y^2}+1\right)^2}{4t}} \frac{\mathrm{d}y}{y} \, \mathrm{d}v.
\end{aligned}
$$

We change variable to $u = \frac{\log\left(\frac{n^2}{y^2}+1\right)}{2\sqrt{t}}$. From equation (4.4.5) we quote $\frac{\mathrm{d}y}{y} \approx -2\sqrt{t}\,\mathrm{d}u$, and we further observe, using once more $n^2+y^2 \approx (n+1)^2$, that

$$
y^{\frac{v}{t}} \approx \left(\frac{(n+1)^2}{\frac{n^2}{y^2}+1}\right)^{\frac{v}{2t}} = (n+1)^{\frac{v}{t}} e^{-\log\left(\frac{n^2}{y^2}+1\right)\frac{v}{2t}} = (n+1)^{\frac{v}{t}} \, e^{-\frac{u\,v}{\sqrt{t}}}.
$$

Then, we deduce

$$
\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \, \frac{\mathrm{d}y}{y^2} \ll \frac{1}{\sqrt{t}} \int_0^1 \frac{e^{-\frac{v^2}{4t}}}{\sqrt{v}} \sum_{n=1}^{\infty} \frac{1}{n+1} \int_{\frac{\log(n^2+1)}{2\sqrt{t}}}^{\infty} u \, e^{-u^2 - \frac{u\,v}{\sqrt{t}}} \mathrm{d}u \, \mathrm{d}v.
$$

We further change the variable of the inner integral to $w = u + \frac{v}{2\sqrt{t}}$, therefore substituting $\mathrm{d}w = \mathrm{d}u$, and we observe

$$
e^{-\frac{v^2}{4t} - u^2 - \frac{u\,v}{\sqrt{t}}} = e^{-w^2}.
$$

Thus, we find

$$
\int_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \, \frac{\mathrm{d}y}{y^2} \ll \frac{1}{\sqrt{t}} \int_0^1 \frac{1}{\sqrt{v}} \sum_{n=1}^{\infty} \frac{1}{n+1} \int_{\frac{\log(n^2+1)+v}{2\sqrt{t}}}^{\infty} \left(w - \frac{v}{2\sqrt{t}}\right) e^{-w^2} \mathrm{d}w \, \mathrm{d}v.
$$

Performing an explicit integration and using the positivity of the complementary error function, we find

$$0 < \int\limits_{\frac{\log(n^2+1)+v}{2\sqrt{t}}}^{\infty} \left(w - \frac{v}{2\sqrt{t}} \right) e^{-w^2} \mathrm{d}w = \frac{e^{-\frac{\left(\log(n^2+1)+v\right)^2}{4t}}}{2} - \mathrm{erfc}\left(\frac{\log(n^2+1)+v}{2\sqrt{t}} \right)$$

$$< e^{-\frac{\left(\log(n^2+1)+v\right)^2}{4t}} < e^{-\frac{\log(n+1)^2}{4t}} \leq e^{-\frac{\log(n+1)^2}{8t} - \frac{\log(2)^2}{8t}}.$$

Therefore, we obtain

$$\int\limits_0^1 \sum_{n=1}^{\infty} A_k^{(0,1)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll \frac{1}{\sqrt{t}} \int\limits_0^1 \frac{1}{\sqrt{v}} \sum_{n=1}^{\infty} \frac{1}{n+1} e^{-\frac{\log(n+1)^2}{8t} - \frac{\log(2)^2}{8t}} \mathrm{d}v$$

$$= \frac{e^{-\frac{\log(2)^2}{8t}}}{\sqrt{t}} \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\frac{\log(n+1)}{8t}+1}} \int\limits_0^1 \frac{\mathrm{d}v}{\sqrt{v}} \ll_k e^{-\frac{1}{20t}},$$

where we used $\frac{\log(n+1)}{8t} + 1 > 1$ uniformly and $\frac{e^{-\frac{\log(2)^2}{8t}}}{\sqrt{t}} \ll_k e^{-\frac{1}{20t}}$, which are valid by the assumption $t \ll_k 1$. This completes the desired estimate for small t for the term (4.4.8).

Estimate of (4.4.9) *for t small and $k = 0$.* Applying the bound (2.3.4), we find

$$\int\limits_0^1 \sum_{n=1}^{\infty} A_0^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll \int\limits_0^1 \sum_{n=1}^{\infty} \frac{e^{-\frac{(t+d(z,n)+1)^2}{4t}}}{\sqrt{t}} \frac{\mathrm{d}y}{y^2}.$$

Let us observe that the assumption $t \leq 1$ implies

$$\frac{(t + d(z,n) + 1)^2}{4t} = \frac{t}{4} + \frac{d(z,n)+1}{2} + \frac{d(z,n)^2 + 2d(z,n) + 1}{4t}$$

$$\geq \frac{d(z,n)}{2} + \frac{2d(z,n)}{4t} + \frac{1}{4t} \geq d(z,n) + \frac{1}{4t}.$$

Further applying the lower bound (4.4.4) for the hyperbolic distance, we obtain

$$\int\limits_0^1 \sum_{n=1}^{\infty} A_0^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll \frac{e^{-\frac{1}{4t}}}{\sqrt{t}} \int\limits_0^1 \sum_{n=1}^{\infty} e^{-d(z,n)} \frac{\mathrm{d}y}{y^2} \ll \frac{e^{-\frac{1}{4t}}}{\sqrt{t}} \int\limits_0^1 \sum_{n=1}^{\infty} \frac{1}{\frac{n^2}{y^2}+1} \frac{\mathrm{d}y}{y^2}$$

$$\ll \frac{e^{-\frac{1}{4t}}}{\sqrt{t}} \int\limits_0^1 \sum_{n=1}^{\infty} \frac{1}{n^2} \mathrm{d}y \ll \frac{e^{-\frac{1}{4t}}}{\sqrt{t}} \ll e^{-\frac{1}{20t}}, \tag{4.4.10}$$

where, for the last inequality, we used the assumption $t \leq 1$.

Estimate of (4.4.9) *for t small and $k \geq 1$.* We remark that we are in the hypothesis of the estimate (2.3.6), because we assumed $t \leq \frac{\log(2)}{2k-1} < \frac{\log(2)+1}{2k-1}$. Applying it, together with the elementary bounds $\left(1 + \frac{1}{\sqrt{t}}\right) \ll \frac{1}{\sqrt{t}}$ and $e^{-(2t+d(z,n))k} < e^{-d(z,n)}$, we find

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll_k \int_0^1 \sum_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt{t}}\right) e^{-(2t+d(z,n))k - \frac{(d(z,n)-t(2k-1)+1)^2}{4t}} \frac{\mathrm{d}y}{y^2}$$
$$\ll_k \frac{1}{\sqrt{t}} \int_0^1 \sum_{n=1}^{\infty} e^{-d(z,n) - \frac{(d(z,n)-t(2k-1)+1)^2}{4t}} \frac{\mathrm{d}y}{y^2}.$$

Since $d(z,n) \geq \log(2)$ and we assumed $t \leq \frac{\log(2)}{2k-1}$, we have the estimate

$$e^{-\frac{(d(z,n)-t(2k-1)+1)^2}{4t}} \leq e^{-\frac{1}{4t}},$$

which implies

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll_k \frac{e^{-\frac{1}{4t}}}{\sqrt{t}} \int_0^1 \sum_{n=1}^{\infty} e^{-d(z,n)} \frac{\mathrm{d}y}{y^2}.$$

Proceeding as in equation (4.4.10) gives the bound

$$\int_0^1 \sum_{n=1}^{\infty} A_k^{(1,\infty)}(t; d(z,n)) \frac{\mathrm{d}y}{y^2} \ll_k e^{-\frac{1}{20t}},$$

which completes the claimed estimate (4.4.7) for t small of $\mathrm{I}_k(t)$. The quantity $\mathrm{I}_k(t)$ is given by the integral of a series with positive and continuous summands. The bounds we just proved show that integral and series converge absolutely, therefore $\mathrm{I}_k(t)$ is well-defined and continuous. □

Observation 4.4.6. For $k \geq 0$ fixed, there exists $\delta_k \in \mathbb{R}_{>0}$ such that, for $0 < y < \delta_k$ and $n \geq 1$, we have the inequality

$$(-1)^k \operatorname{Re}\left(\left(\frac{2iy+n}{2iy-n}\right)^k\right) \geq 0.$$

Proof. By direct computation

$$\operatorname{Re}\left(\left(\frac{2iy+n}{2iy-n}\right)^k\right) = \frac{(-1)^k}{|2iy+n|^{2k}} \operatorname{Re}\left((2iy+n)^{2k}\right)$$
$$= \frac{(-1)^k}{|2iy+n|^{2k}} \sum_{l=0}^{k} \binom{2k}{2l} (-1)^l (2y)^{2l} n^{2k-2l},$$

and the sum in the last expression can be rewritten as

$$n^{2k}+4y^2\left(\sum_{l=1}^{k}\binom{2k}{2l}(-1)^l(2y)^{2l-2}n^{2k-2l}\right),$$

therefore it has fixed sign for $n\geq 1$ and y small enough. $\square$

Proposition 4.4.7. *The term* (APar3) *has the asymptotic expansion*

$$p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}$$

$$=p\int_0^\infty\int_0^1\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\frac{\mathrm{d}y}{y^2}\,\frac{\mathrm{d}t}{t}+o(1)\qquad(\epsilon\to 0)\,,$$

where the integral expression on the right hand side is a well-defined real constant.

Proof. We compute the estimate

$$\left|\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z)\right|$$

$$\leq\int_{\mathcal{F}_\infty(y<1)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left|\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\right|\mu_{\mathrm{hyp}}(z)$$

$$=\int_{\mathcal{F}_\infty(y<1)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z)=\,2\,\mathrm{I}_k(t).$$

By lemma 4.4.5, the latter quantity has exponential decay for $t\to 0$ and for $t\to\infty$. Now, for any $s\in\mathbb{C}$, we have the integral representation

$$\mathcal{M}\left(\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z),s\right)$$

$$=\int_0^\infty\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z)\,t^{s-1}\,\mathrm{d}t.$$

In particular, we can evaluate the s-derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0}$$
$$=\int\limits_0^\infty\int\limits_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z)\,\frac{\mathrm{d}t}{t}.$$

Since the last formula can be rewritten as

$$2\int\limits_0^\infty\int\limits_{\mathcal{F}_\infty(y<1)\setminus B_\infty(\epsilon)}\sum_{n=1}^\infty \mathrm{Re}\left(\left(\frac{2iy+n}{2iy-n}\right)^k\right) K_k(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z)\,\frac{\mathrm{d}t}{t},$$

observation 4.4.6 implies that, for $\epsilon<\delta_k$, the last integral is monotone increasing and decreasing for $\epsilon\to 0$ for k even and odd, respectively. Therefore we can apply the monotone convergence theorem to carry the limit for $\epsilon\to 0$ under the t-integration. This proves the proposition. □

4.5 Degeneration of the hyperbolic contribution to the smoothened determinant

In this section we prove that the term (AHyp) converges, in the limit for $\epsilon\to 0$, to the regularized determinant defined in section 2. We begin by proving an estimate on the integral of the hyperbolic part $\mathrm{H}K_k^\Gamma(t;z,z)$ of the heat kernel on a cusp neighborhood.

Lemma 4.5.1. *Recall the definition of r_Γ from lemma 2.5.1, and let ϵ be small enough such that $-\frac{\log(\epsilon)}{2\pi}>\frac{\sqrt{6}}{r_\Gamma}$. Then*

$$\int\limits_{B_\epsilon(i\infty)}\left|\mathrm{H}K_k^\Gamma(t;z,z)\right|\mu_{\mathrm{hyp}}(z)=O_\Gamma\big(e^{-\frac{1}{9t}}\big)\qquad(t\to 0)\,,$$

and

$$\int\limits_{B_\epsilon(i\infty)}\left|\mathrm{H}K_k^\Gamma(t;z,z)\right|\mu_{\mathrm{hyp}}(z)=\begin{cases}O_\Gamma(t), & k=0,\\ O_\Gamma\big(e^{-\frac{t}{5}}\big), & k\geq 1,\end{cases}\qquad(t\to\infty)\,.$$

Proof. In the first part of the proof we provide an estimate uniform in t for the absolute value of $\mathrm{H}K_k^\Gamma(t;z,z)$ for z close to the cusp $i\infty$. The computation is in the same spirit of the proof of the convergence of the second integral of proposition (2.5.4), but, in order to have a t-uniform estimate, we will lose the decay for y large.
We bound the hyperbolic part by a Stieltjes integral, and we subsequently use lemma 2.5.2, to find

$$
\begin{aligned}
\left|\mathrm{H}K_k^{\Gamma}(t;z,z)\right| &= \left| \sum_{\substack{\gamma\in\Gamma\\ \text{hyperbolic}}} \left(\frac{cz+d}{c\bar{z}+d}\right)^k \left(\frac{\gamma(z)-\bar{z}}{z-\gamma(\bar{z})}\right)^k K_k(t;d_{\mathrm{hyp}}(z,\gamma(z)))\right| \\
&\leq \sum_{\gamma\in\Gamma\backslash\Gamma_\infty} K_k(t;d_{\mathrm{hyp}}(z,\gamma(z))) \leq \int_1^\infty K_k(t;u)\,\mathrm{dN}_{\Gamma\backslash\Gamma_\infty}(z,u) \\
&\leq y \int_{a_\Gamma(z)}^\infty K_k(t;u)\,e^u \mathrm{d}u.
\end{aligned}
$$

We recall that $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right)$. The assumption $-\frac{\log(\epsilon)}{2\pi} > \frac{\sqrt{6}}{r_\Gamma}$ implies $y > \frac{\sqrt{6}}{r_\Gamma}$, which in turn implies $a_\Gamma(z) > \log(3) > 1$. Therefore we can apply the decomposition of the heat kernel given by lemma 2.3.2 with $\delta = 1$, which implies the decomposition

$$
\left|\mathrm{H}K_k^{\Gamma}(t;z,z)\right| \ll_k y \int_{a_\Gamma(z)}^\infty A_k^{(0,1)}(t;u)e^u \mathrm{d}u \tag{4.5.1}
$$

$$
+ y \int_{a_\Gamma(z)}^\infty A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u. \tag{4.5.2}
$$

As in the proof of the previous lemma we have a division in cases of the estimates to be proven, depending on the term (4.5.1) or (4.5.2) under consideration, whether t is small and large, and finally whether $k = 0$ or $k \geq 1$.

Preliminary estimate of the term (4.5.1). Applying the bound given by formula (2.3.3) to the term (4.5.1) we find

$$
\begin{aligned}
y \int_{a_\Gamma(z)}^\infty A_k^{(0,1)}(t;u)e^u \mathrm{d}u &\ll \frac{y\, e^{-tk(k+1)}}{t^{\frac{3}{2}}} \int_{a_\Gamma(z)}^\infty u e^{-\frac{(t+u)^2}{4t}+u} \mathrm{d}u \\
&= \frac{y\, e^{-tk(k+1)}}{t^{\frac{3}{2}}} \int_{a_\Gamma(z)}^\infty u e^{-\left(\frac{u}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} \mathrm{d}u \\
&= \frac{4y\, e^{-tk(k+1)}}{\sqrt{t}} \int_{a_\Gamma(z)}^\infty \left(\frac{u}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right) e^{-\left(\frac{u}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} \\
&\quad + \frac{4y\, e^{-tk(k+1)}}{\sqrt{t}} \frac{\sqrt{t}}{2} \int_{a_\Gamma(z)}^\infty e^{-\left(\frac{u}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} \\
&= \frac{2y\, e^{-tk(k+1)}}{\sqrt{t}} e^{-\left(\frac{a_\Gamma(z)}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} + \sqrt{\pi}\, y\, e^{-tk(k+1)} \operatorname{erfc}\left(\frac{a_\Gamma(z)}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right),
\end{aligned} \tag{4.5.3}
$$

where in the last step we changed variable to $v = \frac{u}{2\sqrt{t}} - \frac{\sqrt{t}}{2}$ and performed an explicit integration.

Estimate of the term (4.5.1) *for t small.* We now assume $t \leq \frac{a_\Gamma(z)}{8}$. Formula (5) of [15] states

$$\operatorname{erfc}(u) \leq e^{-u^2} \qquad (u \in \mathbb{R}_{\geq 0}). \tag{4.5.4}$$

The assumption $t \leq \frac{a_\Gamma(z)}{8}$ implies $\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2} > 0$, therefore we have the estimate

$$\begin{aligned} y \int_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u &\ll \frac{2y\, e^{-tk(k+1)}}{\sqrt{t}} e^{-\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} + \sqrt{\pi}\, y\, e^{-tk(k+1)} e^{-\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} \\ &\ll \left(\frac{1}{\sqrt{t}} + 1\right) y\, e^{-tk(k+1)} e^{-\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} \\ &= \left(\frac{1}{\sqrt{t}} + 1\right) y\, e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{a_\Gamma(z)^2}{8t} - \frac{a_\Gamma(z)^2}{8t} + \frac{a_\Gamma(z)}{2}}. \end{aligned}$$

Since $a_\Gamma(z) > 1$ we deduce $e^{-\frac{a_\Gamma(z)^2}{8t}} \leq e^{-\frac{1}{8t}}$. Moreover, the condition $t \leq \frac{a_\Gamma(z)}{8}$ and the explicit expression of $a_\Gamma(z)$ imply

$$y\, e^{-\frac{a_\Gamma(z)^2}{8t} + \frac{a_\Gamma(z)}{2}} \leq y e^{-\frac{a_\Gamma(z)}{2}} \ll_\Gamma 1.$$

Summing up, the term (4.5.1) admits the bound

$$y \int_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \ll_\Gamma \left(\frac{1}{\sqrt{t}} + 1\right) e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{1}{8t}} \ll_\Gamma e^{-t\left(k+\frac{1}{2}\right)^2 - \frac{1}{9t}} \qquad \left(t \leq \frac{a_\Gamma(z)}{8}\right). \tag{4.5.5}$$

Estimate of the term (4.5.1) *for t large.* Assume $t \geq \frac{a_\Gamma(z)}{8}$ and recall $a_\Gamma(z) > 1$, then we have

$$e^{-\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} \ll 1, \qquad \operatorname{erfc}\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right) \ll 1, \qquad \frac{1}{\sqrt{t}} \ll 1.$$

Using these estimates in equation (4.5.3), the term (4.5.1) admits the bound

$$y \int_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \ll 2y\, e^{-tk(k+1)} + \sqrt{\pi}\, y\, e^{-tk(k+1)} \ll y\, e^{-tk(k+1)} \qquad \left(t \geq \frac{a_\Gamma(z)}{8}\right). \tag{4.5.6}$$

Regarding the term (4.5.2) we first consider the case $k = 0$ for t small and large.

Estimate of the term (4.5.2) *for t small and large and $k = 0$.* Formula (2.3.4) implies

$$y \int_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t; d) e^u \mathrm{d}u \ll y \int_{a_\Gamma(z)}^{\infty} \frac{e^{-\frac{(t+u+1)^2}{4t}}}{\sqrt{t}} e^u \mathrm{d}u < y \int_{a_\Gamma(z)}^{\infty} \frac{e^{-\frac{(t+u)^2}{4t}+u}}{\sqrt{t}} \mathrm{d}u$$

$$= 2y \int_{a_\Gamma(z)}^{\infty} e^{-\left(\frac{u}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)} \frac{\mathrm{d}u}{2\sqrt{t}} = \sqrt{\pi}\, y \operatorname{erfc}\left(\frac{a_\Gamma(z)}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right),$$

where in the last equality we applied the change of variable $v = \frac{u}{2\sqrt{t}} - \frac{\sqrt{t}}{2}$. We are now reduced to the second term of equation (4.5.3). Applying the same argument we obtain the estimates

$$y \int_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t; d) e^u \mathrm{d}u \ll_\Gamma e^{-\frac{t}{4} - \frac{1}{9t}} \qquad \left(t \leq \frac{a_\Gamma(z)}{8}\right), \tag{4.5.7}$$

and

$$y \int_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t; d) e^u \mathrm{d}u \ll y \qquad \left(t \geq \frac{a_\Gamma(z)}{8}\right). \tag{4.5.8}$$

Decomposition of the term (4.5.2) *for t small and large and $k \geq 1$.* Splitting the domain of integration, we find

$$y \int_{a_\Gamma(z)}^{\infty} A_k^{(1,\infty)}(t; u) e^u \mathrm{d}u = y \int_{a_\Gamma(z)}^{\max\{t(2k-1)-1,\, a_\Gamma(z)\}} A_k^{(1,\infty)}(t; u) e^u \mathrm{d}u \tag{4.5.9}$$

$$+ y \int_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t; u) e^u \mathrm{d}u. \tag{4.5.10}$$

Estimate of the subterm (4.5.9) *of the term* (4.5.2) *for t small and large and $k \geq 1$.* Applying formula (2.3.5) we obtain

$$y \int_{a_\Gamma(z)}^{\max\{t(2k-1)-1,\, a_\Gamma(z)\}} A_k^{(1,\infty)}(t; u) e^u \mathrm{d}u \ll_k y \int_{a_\Gamma(z)}^{\max\{t(2k-1)-1,\, a_\Gamma(z)\}} \left(\frac{1}{\sqrt{t}} + 1\right) e^{-2tk - u(k-1)} \mathrm{d}u.$$

Observe that if $t \leq \frac{a_\Gamma(z)+1}{2k-1}$ the last expression is identically zero, therefore we assume $t \geq \frac{a_\Gamma(z)+1}{2k-1}$, which implies $t \gg_k 1$. Thus, we have $\frac{1}{\sqrt{t}} + 1 \ll_k 1$, and the integral (4.5.9) has the bound

$$y \int_{a_\Gamma(z)}^{\max\{t(2k-1)-1,\, a_\Gamma(z)\}} A_k^{(1,\infty)}(t; u) e^u \mathrm{d}u \ll_k y\, t e^{-2tk},$$

because $e^{-u(k-1)} \leq 1$ and the length of the domain of integration is bounded by $t(2k-1) \ll_k t$. Using $a_\Gamma(z) = \log\left(\frac{r_\Gamma^2 y^2}{2}\right)$ and $t \geq \frac{a_\Gamma(z)+1}{2k-1}$ we find

$$y\, te^{-2kt} = y\, te^{-(2k-1)t-t} \leq y\, te^{-a_\Gamma(z)-1-t} \ll_\Gamma \frac{te^{-t}}{y}.$$

We deduce that the term (4.5.9) is estimated by the expression

$$y \int\limits_{a_\Gamma(z)}^{\max\{t(2k-1)-1,\, a_\Gamma(z)\}} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_{\Gamma,k} \begin{cases} 0, & t \leq \frac{a_\Gamma(z)+1}{2k-1}, \\ \frac{te^{-t}}{y}, & t \geq \frac{a_\Gamma(z)+1}{2k-1}. \end{cases} \tag{4.5.11}$$

Preliminary estimate of the subterm (4.5.10) *of the term* (4.5.2) *for t small and large and $k \geq 1$.* For u in the integration domain of the term (4.5.10) we are in the hypothesis of formula (2.3.6). Applying it, we obtain

$$\begin{aligned} y &\int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \\ &\ll_k y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} \left(\frac{1}{\sqrt{t}}+1\right) e^{-2tk-u(k-1)-\frac{(u-t(2k-1)+1)^2}{4t}} \mathrm{d}u. \end{aligned}$$

We observe the relation

$$\begin{aligned} -2tk - u(k-1) - &\frac{(u-t(2k-1)+1)^2}{4t} \\ &= -2tk - u(k-1) - \left(\frac{u+1}{2\sqrt{t}}\right)^2 - t\left(k-\frac{1}{2}\right)^2 + (u+1)\left(k-\frac{1}{2}\right) \\ &= -tk(k+1) - \frac{t}{4} - \left(\frac{u+1}{2\sqrt{t}}\right)^2 + u - \frac{u}{2} + k - \frac{1}{2} \\ &= -tk(k+1) - \left(\frac{u+1}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2 + k - 1. \end{aligned}$$

Further using $e^{k-1} \ll_k 1$, it implies

$$\begin{aligned} y &\int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \\ &\ll_k ye^{-tk(k+1)}\left(\frac{1}{\sqrt{t}}+1\right) \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} e^{-\left(\frac{u+1}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} \mathrm{d}u \\ &= 2ye^{-tk(k+1)}\left(\sqrt{t}+1\right) \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} e^{-\left(\frac{u+1}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right)^2} \frac{\mathrm{d}u}{2\sqrt{t}} \end{aligned}$$

$$= \frac{\sqrt{\pi}\, y e^{-tk(k+1)}}{4} \left(\sqrt{t}+1\right) \operatorname{erfc}\left(\frac{\max\{t(2k-1)-1,\, a_\Gamma(z)\}+1}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right). \qquad (4.5.12)$$

Estimate of the subterm (4.5.10) *of the term* (4.5.2) *for t small and $k \geq 1$.* Assuming $t \leq \frac{a_\Gamma(z)+1}{2k-1}$, the last expression (4.5.12) reduces to

$$y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_k y e^{-tk(k+1)} \left(\sqrt{t}+1\right) \operatorname{erfc}\left(\frac{a_\Gamma(z)+1}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right).$$

Since $t \leq \frac{a_\Gamma(z)+1}{2k-1} \leq a_\Gamma(z)+1$, we can apply the inequality (4.5.4) and obtain

$$\begin{aligned} y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u &\ll_k y e^{-tk(k+1)} \left(\sqrt{t}+1\right) e^{-\left(\frac{a_\Gamma(z)+1}{2\sqrt{t}} - \frac{\sqrt{t}}{2}\right)^2} \\ &= y e^{-tk(k+1)} \left(\sqrt{t}+1\right) e^{-\frac{a_\Gamma(z)^2}{4t} - \frac{a_\Gamma(z)}{2t} - \frac{1}{4t} + \frac{a_\Gamma(z)+1}{2} - \frac{t}{4}} \\ &\ll_k y e^{-\frac{t}{4} - \frac{1}{4t}} \left(\sqrt{t}+1\right) e^{-t\left(\frac{a_\Gamma(z)^2}{4t^2} - \frac{a_\Gamma(z)}{2t} + k(k+1)\right)}, \end{aligned}$$

where in the last estimate we used $e^{-\frac{a_\Gamma(z)}{2t} + \frac{1}{2}} \ll 1$. We introduce the variable

$$\alpha_\Gamma(t,z) := \frac{a_\Gamma(z)}{t},$$

and we observe

$$y \approx_\Gamma \frac{r_\Gamma y}{\sqrt{2}} = e^{\frac{a_\Gamma(z)}{2}} = e^{t\,\frac{\alpha_\Gamma(t,z)}{2}}.$$

Further using $k(k+1) \geq 2$, we find

$$y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_k e^{-\frac{t}{4} - \frac{1}{4t}} \left(\sqrt{t}+1\right) e^{-t\left(\frac{\alpha_\Gamma(t,z)^2}{4} - \alpha_\Gamma(t,z) + 2\right)}.$$

The real polynomial $\frac{v^2}{4} - v + 2$ has minimum 1 attained at $v = 2$, therefore

$$e^{-t\left(\frac{\alpha_\Gamma(t,z)^2}{4} - \alpha_\Gamma(t,z) + 2\right)} \leq e^{-t} \leq 1.$$

Thus the term (4.5.10) admits the estimate

$$y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_k e^{-\frac{t}{4} - \frac{1}{4t}} \left(\sqrt{t}+1\right) \ll_k e^{-\frac{t}{5} - \frac{1}{5t}}$$

$$\left(t \leq \frac{a_\Gamma(z)+1}{2k-1}\right). \qquad (4.5.13)$$

Estimate of the subterm (4.5.10) *of the term* (4.5.2) *for t large and $k \geq 1$.* Assuming $t \geq \frac{a_\Gamma(z)+1}{2k-1}$, using the elementary bound $\mathrm{erfc}(u) \leq 1$ valid for $u \in \mathbb{R}$, the estimate given by equation (4.5.12) reduces to

$$
\begin{aligned}
y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} & A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \\
&\ll_k ye^{-tk(k+1)}\left(\sqrt{t}+1\right)\mathrm{erfc}\left(\frac{t(2k-1)-1+1}{2\sqrt{t}}-\frac{\sqrt{t}}{2}\right) \\
&\ll_k ye^{-tk(k+1)}\left(\sqrt{t}+1\right).
\end{aligned}
$$

Since $a_\Gamma(z) > 1$, we have $t \geq \frac{a_\Gamma(z)+1}{2k-1} \gg_k 1$ and $\sqrt{t}+1 \ll_k \sqrt{t}$. Moreover, $k \geq 1$ implies $k(k+1) \geq (2k-1)+1$, therefore

$$
\begin{aligned}
y \int\limits_{\max\{t(2k-1)-1,\, a_\Gamma(z)\}}^{\infty} & A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_k y\, e^{-t(2k-1)-t}\sqrt{t} \\
&\ll_k y\, e^{-a_\Gamma(z)-1-t}\sqrt{t} \ll_{\Gamma,k} \frac{e^{-t}}{y}\sqrt{t} \ll_{\Gamma,k} \frac{e^{-\frac{t}{2}}}{y} \\
&\left(t \geq \frac{a_\Gamma(z)+1}{2k-1}\right). \qquad (4.5.14)
\end{aligned}
$$

The combination of the estimates (4.5.11), (4.5.13) and (4.5.14) provides a bound for the term (4.5.2) for $k \geq 1$, given by

$$
y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \ll_{\Gamma,k} \begin{cases} e^{-\frac{t}{5}-\frac{1}{5t}}, & t \leq \frac{a_\Gamma(z)+1}{2k-1}, \\ \frac{e^{-\frac{t}{2}}}{y}, & t \geq \frac{a_\Gamma(z)+1}{2k-1}. \end{cases} \qquad (4.5.15)
$$

This completes the proof of the preliminary estimates, we now use them to prove the lemma. The bound of $\left|\mathrm{H}K_k^\Gamma(t;z,z)\right|$ in terms of (4.5.1) and (4.5.2) implies

$$
\begin{aligned}
\int\limits_{B_\epsilon(i\infty)} \left|\mathrm{H}K_k^\Gamma(t;z,z)\right| \mu_{\mathrm{hyp}}(z) &= \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} \left|\mathrm{H}K_k^\Gamma(t;z,z)\right| \frac{\mathrm{d}y}{y^2} \\
&\ll_{\Gamma,k} \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} \qquad (4.5.16) \\
&+ \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(1,\infty)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2}. \qquad (4.5.17)
\end{aligned}
$$

Estimate of the term (4.5.16) *for t small.* Without loss of generality we assume $t \leq \frac{\log(3)}{8} \leq \frac{a_\Gamma(z)}{8}$. Therefore we can apply the bound (4.5.5) and obtain

$$\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} \ll_\Gamma \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} e^{-t\left(k+\frac{1}{2}\right)^2-\frac{1}{9t}} \frac{\mathrm{d}y}{y^2} = O_\Gamma\left(e^{-\frac{1}{9t}}\right) \qquad (t \to 0)\,.$$

Estimate of the term (4.5.16) *for t large.* Since the domain of integration of the y-integral is not bounded from above, and therefore $a_\Gamma(z)$ is not bounded from above, even though t is large we still have the two cases $t \leq \frac{a_\Gamma(z)}{8}$ and $t \geq \frac{a_\Gamma(z)}{8}$. Observe that $t \leq \frac{a_\Gamma(z)}{8}$ is equivalent to $y \geq \frac{\sqrt{2}e^{4t}}{r_\Gamma}$. Then, we split the domain of the y-integration at $\frac{\sqrt{2}e^{4t}}{r_\Gamma}$ and apply the bounds (4.5.6) and (4.5.5) to the two resulting integrals, respectively. We obtain

$$\begin{aligned}
&\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} \\
&\qquad = \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\frac{\sqrt{2}e^{4t}}{r_\Gamma}} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} + \int\limits_{\frac{\sqrt{2}e^{4t}}{r_\Gamma}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(0,1)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} \\
&\qquad \ll_\Gamma \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\frac{\sqrt{2}e^{4t}}{r_\Gamma}} y\, e^{-tk(k+1)} \frac{\mathrm{d}y}{y^2} + \int\limits_{\frac{\sqrt{2}e^{4t}}{r_\Gamma}}^{\infty} e^{-t\left(k+\frac{1}{2}\right)^2-\frac{1}{9t}} \frac{\mathrm{d}y}{y^2} \\
&\qquad = e^{-tk(k+1)}\left(\log\left(\frac{\sqrt{2}e^{4t}}{r_\Gamma}\right) - \log\left(-\frac{\log(\epsilon)}{2\pi}\right)\right) + O_\Gamma\left(e^{-t\left(k+\frac{1}{2}\right)^2}\right) \\
&\qquad \ll e^{-tk(k+1)}\left(4t + \log\left(\frac{\sqrt{2}}{r_\Gamma}\right)\right) + O_\Gamma\left(e^{-t\left(k+\frac{1}{2}\right)^2}\right) \\
&\qquad = O_\Gamma\left(te^{-tk(k+1)}\right) \qquad (t \to \infty)\,.
\end{aligned}$$

As in the corresponding case in the proof of the preliminary estimates, we first consider the term (4.5.17) for $k = 0$ and t small and large.

Estimate of the term (4.5.17) *for t small and large and $k = 0$.* The bound (4.5.7) implies the asymptotic expansion for t small given by

$$\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t;u)e^u \mathrm{d}u \frac{\mathrm{d}y}{y^2} = O_\Gamma\left(e^{-\frac{1}{9t}}\right) \qquad (t \to 0)\,.$$

For the expansion for t large we proceed along the lines of of the computation just done to estimate the term (4.5.16). We split the y-integration domain at $\frac{\sqrt{2}e^{4t}}{r_\Gamma}$ to obtain

$$\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2} = \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\frac{\sqrt{2}e^{4t}}{r_\Gamma}} y \int\limits_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2} + \int\limits_{\frac{\sqrt{2}e^{4t}}{r_\Gamma}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2}.$$

We now apply the bounds (4.5.8) and (4.5.7) to the first and second term, respectively. We find

$$\begin{aligned}\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_0^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2} &\ll \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\frac{\sqrt{2}e^{4t}}{r_\Gamma}} \frac{\mathrm{d}y}{y} + \int\limits_{\frac{\sqrt{2}e^{4t}}{r_\Gamma}}^{\infty} e^{-\frac{t}{4}-\frac{1}{9t}}\frac{\mathrm{d}y}{y^2}\\ &= \log\left(\frac{\sqrt{2}e^{4t}}{r_\Gamma}\right) - \log\left(-\frac{\log(\epsilon)}{2\pi}\right) + O_\Gamma\big(e^{-\frac{t}{4}}\big)\\ &= O_\Gamma(t) \qquad (t\to\infty)\,.\end{aligned}$$

Estimate of the term (4.5.17) *for t small and large and $k \geq 1$.* For t small formula (4.5.15) implies

$$\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2} \ll_k \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} e^{-\frac{t}{5}-\frac{1}{5t}}\frac{\mathrm{d}y}{y^2} = O_\Gamma\left(e^{-\frac{1}{5t}}\right) \qquad (t\to 0)\,,$$

and for t large the same formula implies

$$\int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} y \int\limits_{a_\Gamma(z)}^{\infty} A_k^{(1,\infty)}(t;u)e^u\mathrm{d}u\frac{\mathrm{d}y}{y^2} \ll_{\Gamma,k} \int\limits_{-\frac{\log(\epsilon)}{2\pi}}^{\infty} e^{-\frac{t}{2}}\frac{\mathrm{d}y}{y^3} = O_{\Gamma,k}\left(e^{-\frac{t}{5}}\right) \qquad (t\to\infty)\,.$$

This completes the proof of the lemma. □

Proposition 4.5.2. *The term* (AHyp) *has the asymptotic expansion*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(\mathrm{H}K_k^\Gamma(t;z,z)+K_k(t;0)\right)\mu_{\mathrm{hyp}}(z),s\right)\right)_{s=0}\\ = -\log\left(\det\nolimits_\Gamma^*(\Delta_k)\right)+o(1) \qquad (\epsilon\to 0)\,.\end{aligned}$$

Proof. Recalling the definition 2.6.1 of the regularized determinant, and using the linearity of the Mellin transform, we compute

$$-\log\left(\det\nolimits^{*}_{\Gamma}(\Delta_k)\right)-\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\bigg(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\big(\mathrm{H}K_k^{\Gamma}(t;z,z)+K_k(t;0)\big)\,\mu_{\mathrm{hyp}}(z),s\bigg)\right)\Bigg|_{s=0}$$

$$=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}$$

$$+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)}K_k(t;0)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}.$$

We will show that both terms on the right hand side vanish in the limit for $\epsilon\to 0$. By formula (2.8.7), we have

$$\lim_{\epsilon\to 0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)}K_k(t;0)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}=\lim_{\epsilon\to 0}c_k\sum_{j=1}^{p}\mathrm{vol}_{\mathrm{hyp}}(B_\epsilon(P_j))=0.$$

To examine the term involving the hyperbolic part of $K_k^{\Gamma}(t;z,z)$ we first reduce, using the fact that scaling matrices are isometries, to the cusp $i\infty$. We have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}$$

$$=p\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}.$$

Let $k\geq 1$. Since

$$\left|\int_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z)\right|\leq\int\limits_{B_\epsilon(i\infty)}\left|\mathrm{H}K_k^{\Gamma}(t;z,z)\right|\,\mu_{\mathrm{hyp}}(z),$$

we can apply lemma (4.5.1) and conclude that

$$\mathcal{M}\left(\int_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)=\int\limits_{0}^{\infty}\int\limits_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z)\,t^{s-1}\mathrm{d}t$$

is an entire function in s. We use the absolute convergence to carry the s-derivative under the integral sign. Moreover, formula (2.8.2) implies the evaluation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{t^s}{\Gamma(s)}\right)\Bigg|_{s=0}=\lim_{s\to 0}\left(\frac{\log(t)t^s}{\Gamma(s)}\right)-\lim_{s\to 0}\left(\frac{\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)\,t^s}{\Gamma(s)^2}\right)=1.$$

Thus, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z),s\right)\right)\Bigg|_{s=0}=\int\limits_{0}^{\infty}\int\limits_{B_\epsilon(i\infty)}\mathrm{H}K_k^{\Gamma}(t;z,z)\,\mu_{\mathrm{hyp}}(z)\,\frac{\mathrm{d}t}{t}.$$

For any $t > 0$ fixed, the quantity

$$\int_{B_\epsilon(i\infty)} \mathrm{H}K_k^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z)$$

is strictly monotone decreasing to 0 for $\epsilon \to 0$. Therefore, by the monotone convergence theorem, we can carry the limit for $\epsilon \to 0$ under the t-integration and obtain

$$\lim_{\epsilon \to 0} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_k^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right) \right)\Bigg|_{s=0} = 0.$$

This completes the proof of the proposition for $k \geq 1$. Regarding the case $k = 0$, lemma 4.5.1 only implies the integral representation

$$\mathcal{M} \left(\int_{B_\epsilon(i\infty)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right) = \int_0^\infty \int_{-\frac{\log(\epsilon)}{2\pi}}^\infty \mathrm{H}K_0^\Gamma(t; z, z)\, \frac{\mathrm{d}y}{y^2}\, t^{s-1} \mathrm{d}t \qquad (\mathrm{Re}(s) < -1)\,. \tag{4.5.18}$$

The holomorphicity of the term

$$\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right)$$

is thus only proven for $\mathrm{Re}(s) < -1$. By linearity of the Mellin transform, we have the equality

$$\begin{aligned}
\frac{1}{\Gamma(s)} \mathcal{M} &\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right) \\
&= \frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\mathcal{F}_\Gamma} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right) \\
&\quad - \frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right)
\end{aligned}$$

for any $s \in \mathbb{C}$ such that one of the sides is defined. We now use this relation to prove that

$$\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right)$$

is an entire function in s. The holomorphicity of

$$\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\mathcal{F}_\Gamma} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right) = \frac{1}{\Gamma(s)} \mathcal{M} \left(\mathrm{HTr}\, K_0^\Gamma(t), s \right)$$

for every $s \in \mathbb{C}$ has been proved in the second chapter of this work, and it follows from an application of the direct mapping theorem for the Mellin transform, theorem (B.2), that uses lemmata 2.6.2 and 2.6.3. Also the entirety of

$$\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t; z, z)\, \mu_{\mathrm{hyp}}(z), s \right)$$

will follow, after an analysis of the asymptotic expansions for t small and for t large of its argument, by an application of the direct mapping theorem for the Mellin transform. Lemmata 2.6.2 and 4.5.1 imply the relation

$$\int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) = \mathrm{HTr}\,K_0^\Gamma(t) - \int\limits_{\bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z)$$
$$= O_\Gamma\big(e^{-\frac{b_\Gamma}{t}}\big) + p\,O_\Gamma\big(e^{-\frac{1}{9t}}\big) \qquad (t{\to}0)\,,$$

which proves the exponential decay for $t{\to}0$. To examine the asymptotic expansion for t large we decompose the hyperbolic part of the heat kernel according to formula (2.4.1) and definition 2.4.1. We find

$$\int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) = \int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) \tag{4.5.19}$$
$$- \int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} K_0(t;0)\,\mu_{\mathrm{hyp}}(z) \tag{4.5.20}$$
$$- \int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{P}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z). \tag{4.5.21}$$

From lemma 2.6.3 we recall the notation $d_{\Gamma,0} = \min\big\{\frac{1}{4}, \lambda_{1,\Gamma,0}\big\}$, where $\lambda_{1,\Gamma,0}$ is the first non-zero eigenvalue of $\Delta_0 = D_0$. Then the spectral expansion of the heat kernel, given on page 276 of [46], implies

$$K_0^\Gamma(t;z,z) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O_z\big(e^{-td_{\Gamma,0}}\big) \qquad (t{\to}\infty)\,. \tag{4.5.22}$$

Taking the maximum of the implicit constant for z ranging over the compact domain of integration of the term (4.5.19) gives the asymptotic expansion

$$\int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) = \frac{\mathrm{vol}_{\mathrm{hyp}}\Big(X \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)\Big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O_\epsilon\big(e^{-td_{\Gamma,0}}\big) \qquad (t{\to}\infty)\,.$$

Also the term (4.5.20) has exponential decay for t large, since formula (2.2.5) implies

$$K_0(t;0) = O\big(e^{-\frac{t}{4}}\big) \qquad (t{\to}\infty)\,.$$

To bound the remaining term (4.5.21) we explicit the definition of the parabolic part

$$\int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{P}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z)$$
$$= \int\limits_{\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)} \sum_{\gamma\in P(\Gamma)} \sum_{[\eta]\in\Gamma_\gamma\setminus\Gamma} \sum_{n=1}^{\infty} K_0(t; d_{\mathrm{hyp}}(z, \eta^{-1}\gamma^n\eta(z)))\,\mu_{\mathrm{hyp}}(z).$$

We unfold the domain of integration along the cosets $[\eta] \in \Gamma_\gamma \backslash \Gamma$. Let $\gamma_h \in P(\Gamma)$ be a generator of the stabilizer of the cusp P_h. We have the inclusion

$$\sigma_h^{-1}\left(\bigcup_{[\eta]\in\Gamma_{\gamma_h}\backslash\Gamma} \eta\left(\mathcal{F}_\Gamma \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)\right)\right) \subset \mathcal{F}_\infty\left(y < -\frac{\log(\epsilon)}{2\pi}\right).$$

Since the integrand is strictly positive, the inclusion of domains of integration provides the bound

$$\int\limits_{\mathcal{F}_\Gamma\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{P}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) \leq p \int\limits_{\mathcal{F}_\infty\left(y<-\frac{\log(\epsilon)}{2\pi}\right)} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} K_0(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z).$$

Mimicking what we already did for the decomposition of the regularized determinant, we split the domain of integration at $y=1$ and we add and subtract the identity component. We thus obtain

$$\begin{aligned}\int\limits_{\mathcal{F}_\Gamma\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \mathrm{P}K_0^\Gamma(t;z,z)\,\mu_{\mathrm{hyp}}(z) \leq{}& p \int\limits_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} \sum_{n\in\mathbb{Z}} K_0(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z)\\ &+ p \int\limits_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} K_0(t;0)\,\mu_{\mathrm{hyp}}(z)\\ &+ p \int\limits_{\mathcal{F}_\infty(y<1)} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} K_0(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z).\end{aligned}$$

The integrand of the first summand in the last decomposition is the left hand side of theorem 3.4.1 for $k=0$. Applying it, we deduce

$$\sum_{n\in\mathbb{Z}} K_0(t;d_{\mathrm{hyp}}(z,z+n)) = O_z\big(e^{-\frac{t}{4}}\big) \qquad (t\to\infty).$$

Taking the maximum of the implicit constant for z ranging over the compact domain of integration, we find

$$\int\limits_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} \sum_{n\in\mathbb{Z}} K_0(t;d_{\mathrm{hyp}}(z,z+n))\,\mu_{\mathrm{hyp}}(z) = O_\epsilon\big(e^{-\frac{t}{4}}\big) \qquad (t\to\infty).$$

By formula (2.2.5), we have

$$K_0(t;0) = O\big(e^{-\frac{t}{4}}\big) \qquad (t\to\infty),$$

and, since the domain of integration $\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)$ has volume bounded by 1, we find

$$\int\limits_{\mathcal{F}_\infty\left(1<y<-\frac{\log(\epsilon)}{2\pi}\right)} K_0(t;0)\,\mu_{\text{hyp}}(z)=O\big(e^{-\frac{t}{4}}\big)\qquad(t\to\infty)\,.$$

The last term has been already examined in the study of (APar3), and, by lemma 4.4.5, it has exponential decay for t large

$$\int\limits_{\mathcal{F}_\infty(y<1)}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}K_0(t;d_{\text{hyp}}(z,z+n))\,\mu_{\text{hyp}}(z)=O\big(e^{-\frac{t}{5}}\big)\qquad(t\to\infty)\,.$$

Summing up, we have shown that

$$\int\limits_{\mathcal{F}_\Gamma\setminus\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z)=O_\epsilon\big(e^{-t\,\min\left\{d_{\Gamma,0},\frac{1}{5}\right\}}\big).$$

Therefore, since the argument of the Mellin transform has exponential decay for $t\to0$ and for $t\to\infty$, for any $\epsilon>0$ the function

$$\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\mathcal{F}_\Gamma\setminus\bigcup_{j=1}^p B_\epsilon(P_j)}\mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z),s\right)$$

is entire in s. We can finally conclude that

$$\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z),s\right)$$

is also entire in s for any $\epsilon>0$. Since it is holomorphic it is analytic, and we can consider its Taylor series centered at $s=-2$

$$\begin{aligned}\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z),s\right)&=p\,\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{B_\epsilon(i\infty)}\mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z),s\right)\\&=p\sum_{l=0}^\infty m_{\Gamma,l}(\epsilon)(s+2)^l.\end{aligned}$$

The integral representation (4.5.18) for the holomorphic function in consideration is valid and absolutely convergent for $\mathrm{Re}(s)<-1$. Using it, the Taylor coefficients $m_{\Gamma,l}(\epsilon)$ are given by the formula

$$\begin{aligned}m_{\Gamma,l}(\epsilon)&=\frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}s^l}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{B_\epsilon(i\infty)}\mathrm{H}K_0^\Gamma(t;z,z)\,\mu_{\text{hyp}}(z),s\right)\right)_{s=-2}\\&=\frac{1}{l!}\frac{\mathrm{d}^l}{\mathrm{d}s^l}\left(\frac{1}{\Gamma(s)}\int_0^\infty\int_{-\frac{\log(\epsilon)}{2\pi}}^\infty\mathrm{H}K_0^\Gamma(t;z,z)\,\frac{\mathrm{d}y}{y^2}\,t^{s-1}\mathrm{d}t\right)_{s=-2}\end{aligned}$$

$$= \frac{1}{l!} \int_0^\infty \int_{-\frac{\log(\epsilon)}{2\pi}}^\infty \mathrm{H}K_0^\Gamma(t;z,z)\, \frac{\mathrm{d}y}{y^2}\, \frac{\mathrm{d}^l}{\mathrm{d}s^l} \left(\frac{t^{s-1}}{\Gamma(s)} \right)_{s=-2} \mathrm{d}t.$$

Observe that

$$\frac{\mathrm{d}^l}{\mathrm{d}s^l} \left(\frac{t^{s-1}}{\Gamma(s)} \right)_{s=-2} = \sum_{n=0}^{l} \binom{l}{n} \frac{\mathrm{d}^n}{\mathrm{d}s^n} \left(t^{s-1} \right)_{s=-2} \frac{\mathrm{d}^{l-n}}{\mathrm{d}s^{l-n}} \left(\frac{1}{\Gamma(s)} \right)_{s=-2} = \sum_{n=0}^{l} \frac{l!}{n!} \frac{\log(t)^n}{t^3}\, g_{l-n},$$

where g_{l-n} is the $(l-n)$-th term of the Taylor series of $\frac{1}{\Gamma(s)}$ centered at $s=-2$. Using this evaluation and splitting the domain of integration at $t=1$, we find

$$\begin{aligned} m_{\Gamma,l}(\epsilon) = &\sum_{n=0}^{l} \frac{g_{l-n}}{n!} \int_0^1 \int_{-\frac{\log(\epsilon)}{2\pi}}^\infty \mathrm{H}K_0^\Gamma(t;z,z)\, \frac{\mathrm{d}y}{y^2} \log(t)^n \frac{\mathrm{d}t}{t^3} \\ &+ \sum_{n=0}^{l} \frac{g_{l-n}}{n!} \int_1^\infty \int_{-\frac{\log(\epsilon)}{2\pi}}^\infty \mathrm{H}K_0^\Gamma(t;z,z)\, \frac{\mathrm{d}y}{y^2} \log(t)^n \frac{\mathrm{d}t}{t^3}. \end{aligned}$$

This is a finite sum of absolutely convergent integrals, since each of them is convergent, by lemma (4.5.1), and its integrand has fixed sign. Moreover, the positivity of $\mathrm{H}K_0^\Gamma(t;z,z)$ implies that, for $t>0$ fixed, the function

$$\int_{-\frac{\log(\epsilon)}{2\pi}}^\infty \mathrm{H}K_0^\Gamma(t;z,z)\, \frac{\mathrm{d}y}{y^2}$$

is strictly monotone decreasing to 0 for $\epsilon \to 0$. Therefore, using the monotone convergence theorem, we deduce

$$\lim_{\epsilon \to 0} m_{\Gamma,l}(\epsilon) = 0$$

for every $l \in \mathbb{N}$. Finally, the entirety of

$$\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\, \mu_{\mathrm{hyp}}(z), s \right)$$

implies that its Taylor series converges absolutely in the whole complex plane, specifically at $s=0$. We then take the s-derivative into the series to obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \left(\frac{1}{\Gamma(s)} \mathcal{M} \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \mathrm{H}K_0^\Gamma(t;z,z)\, \mu_{\mathrm{hyp}}(z), s \right) \right)_{s=0} &= p \frac{\mathrm{d}}{\mathrm{d}s} \left(\sum_{l=1}^\infty l\, m_{\Gamma,l}(\epsilon)(s+2)^l \right)_{s=0} \\ &= 2p \sum_{l=1}^\infty l^2\, m_{\Gamma,l}(\epsilon). \end{aligned}$$

The series on the right hand side is the Taylor series of the derivative of an entire function, therefore it is absolutely convergent. Taking the limit for $\epsilon \to 0$ into the series we conclude it is 0. This proves the statement of the proposition for $k=0$. □

4.6 Statement of the regularized arithmetic Riemann–Roch theorem

In this section we combine the results obtained via metric degeneration to state a version of the arithmetic Riemann–Roch theorem valid up to an implicit constant. First, we define the constants $\widetilde{C}(k)$ by summing up the constant terms in the asymptotic expansions of the various contributions to the term (APar). Specifically, adding the constants, without the factor p, from propositions 4.4.1, 4.4.3 and 4.4.7, we define

$$\widetilde{C}(k) := \left(k+\frac{1}{2}\right)\log(2\pi) - \sum_{j=0}^{k-1}\frac{(2k-2j-1)\log((2k-j)(j+1))}{\Gamma(2k-j)\,\Gamma(j+1)}\sum_{n=1}^{\infty}\int_{4\pi n}^{\infty} W_{k,k-j-\frac{1}{2}}(v)^2\frac{\mathrm{d}v}{v^2}$$
$$- c_k + \int_0^{\infty}\int_0^1 \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(\frac{2iy+n}{2iy-n}\right)^k K_k(t;d_{\mathrm{hyp}}(z,z+n))\frac{\mathrm{d}y}{y^2}\,\frac{\mathrm{d}t}{t}.$$

Then, in view of the asymptotic expansion of the smoothened determinant given in corollary 4.3.1, the definition of the terms (B) and (C), and the known partial asymptotics on the smoothened determinant, i.e., propositions 4.4.1, 4.4.3, 4.4.7 and 4.5.2, we define the finite quantities $\widetilde{C}(\Gamma,k)$ by the expression

$$\begin{aligned}\widetilde{C}(\Gamma,k) := \lim_{\epsilon\to 0}\Bigg(& p\,\mathrm{Rest}_{\mathrm{par}}(\epsilon)\\
&+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(K^{\Gamma}_{k,\epsilon}(t;z,z)-K^{\Gamma}_{k}(t;z,z)\right)\mu_\epsilon(z),s\right)\right)_{s=0}\\
&+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^{p}B_\epsilon(P_j)}K^{\Gamma}_{k,\epsilon}(t;z,z)\,\mu_\epsilon(z),s\right)\right)_{s=0}\\
&+\frac{p}{6}\left(\log(\epsilon)-\log(-\log(\epsilon))\right)\Bigg).\end{aligned}\tag{4.6.1}$$

Combining these two constants, and taking into account the factor $\frac{p}{3}\log(2)$, which will be the contribution of the difference of renormalizations of determinants of Laplacians, we define

$$C(\Gamma,k) := -\frac{1}{2}\widetilde{C}(\Gamma,k) - \frac{p}{6}\left(3\,\widetilde{C}(k)+\log(2)\right)$$

We can now state the desired regularized arithmetic Riemann–Roch theorem.

Theorem 4.6.1. *Let $f\colon\mathcal{X}\to\mathcal{S}$ be an arithmetic surface over $\mathcal{S}$ such that $X=\mathcal{X}_{\mathbb{C}}\simeq X(\Gamma)$ for Γ a cofinite torsion-free and discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$, and let $\overline{\mathcal{S}}_{k+1}$ be a hermitian line bundle on $\mathcal{X}$ such that the induced complex hermitian line bundle $\overline{\mathcal{S}}_{k+1,\mathbb{C}}$ is isometric to the line bundle of cusp forms $\overline{S}_{k+1}$ on X. Then, we have the equality of real numbers*

$$\widehat{\deg}\Big(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet}}\Big)+\widehat{\deg}\Big(\mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet}}\Big)+\frac{1}{2}\log\Big(\mathrm{det}^{*}_{\Gamma}\Big(\Delta^{1}_{\overline{S}_{k+1}}\Big)\Big)+\delta_f+C(\Gamma,k)$$
$$=\frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1}.\overline{\mathcal{S}}_{k+1}-6\overline{\mathcal{S}}_{k+1},\overline{\omega}_{\mathcal{X}}+\overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right)^{*}+\frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1(\overline{\omega}_X).$$

Proof. By definition 4.2.5, the right hand side of formula (4.1.1), i.e., the arithmetic Riemann–Roch theorem for the ϵ-regularized Petersson and hyperbolic metrics, has the asymptotic expansion

$$\frac{p}{12}\log(\epsilon)+p\left(\frac{k}{2}+\frac{1}{6}\right)\log(-\log(\epsilon))+\frac{1}{12}\left(6\,\overline{\mathcal{S}}_{k+1}.\overline{\mathcal{S}}_{k+1}-6\overline{\mathcal{S}}_{k+1},\overline{\omega}_{\mathcal{X}}+\overline{\omega}_{\mathcal{X}}.\overline{\omega}_{\mathcal{X}}\right)^{*}$$
$$+\frac{2\zeta'(-1)+\zeta(-1)}{2}\int_X \mathrm{c}_1(\overline{\omega}_X)+o(1)\qquad(\epsilon\to 0)\,.$$

On the other hand, the left hand side of formula (4.1.1) is

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet},\epsilon}\right)+\widehat{\deg}\left(\mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet},\epsilon}\right)+\frac{1}{2}\log\Big(\det{}'\Big(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\Big)\Big)+\delta_f.$$

By formulae (4.1.2) and (4.1.3), we have

$$\widehat{\deg}\left(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet},\epsilon}\right)+\widehat{\deg}\left(\mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet},\epsilon}\right)$$
$$=\widehat{\deg}\Big(\det \mathrm{H}^0(\mathcal{X},\mathcal{S}_{k+1})_{\mathrm{Pet}}\Big)+\widehat{\deg}\Big(\mathrm{H}^1(\mathcal{X},\mathcal{S}_{k+1})^{\vee}_{\mathrm{Pet}}\Big)+o(1)\qquad(\epsilon\to 0)\,.$$

By abuse of notation, we denote in this proof the terms of the decomposition of the smoothened determinant by the number of the defining equation, as in formula (4.3.2). By proposition 4.3.4 and the above mentioned decomposition

$$\frac{1}{2}\log\Big(\det{}'\Big(\Delta^{1}_{\overline{S}_{k+1,\epsilon}}\Big)\Big)=\frac{1}{2}\log\left(\det{}'\left(\Delta_{k,\epsilon}\right)\right)+\left(\frac{(3k+1)(2g-2)}{6}+\frac{p\,k}{2}+N_k\right)\log(2)$$
$$=-\frac{1}{2}\Big((\mathrm{APar1})+(\mathrm{APar2})+(\mathrm{APar3})+(\mathrm{AHyp})+(\mathrm{B})+(\mathrm{C})\Big)$$
$$+\left(\frac{(2g-2)(3k+1)}{6}+\frac{p\,k}{2}+N_k\right)\log(2).$$

We sum up the various formulae proved in this section. By proposition 4.5.2, combined with definition 2.6.5 and observation 2.6.6, we find

$$-\frac{1}{2}(\mathrm{AHyp})+\left(\frac{(2g-2)(3k+1)}{6}+\frac{p\,k}{2}+N_k\right)\log(2)$$
$$=\frac{1}{2}\log\left(\mathrm{det}^{*}_{\Gamma}\left(\Delta_k\right)\right)+\left(\frac{(2g-2)(3k+1)}{6}+\frac{p\,k}{2}+N_k\right)\log(2)+o(1)$$

$$
\begin{aligned}
&= \frac{1}{2}\log\left(\det\nolimits^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)\right) - \left(\frac{(2g-2+p)\,(3k+1)}{6} + N_k\right)\log(2) \\
&\quad + \left(\frac{(2g-2)(3k+1)}{6} + \frac{p\,k}{2} + N_k\right)\log(2) + o(1) \\
&= \frac{1}{2}\log\left(\det\nolimits^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)\right) - \frac{p}{6}\log(2) + o(1) \qquad (\epsilon\to 0)\,.
\end{aligned}
$$

While, by propositions 4.4.1, 4.4.3 and 4.4.7, and the definitions of $\widetilde{C}(k)$ and $\widetilde{C}(\Gamma, k)$, we have

$$
\begin{aligned}
&-\frac{1}{2}\Big((\mathrm{APar1}) + (\mathrm{APar2}) + (\mathrm{APar3}) + (\mathrm{B}) + (\mathrm{C})\Big) \\
&\qquad = \frac{p}{12}\log(\epsilon) + p\left(\frac{k}{2} + \frac{1}{6}\right)\log(-\log(\epsilon)) - \frac{1}{2}\widetilde{C}(\Gamma, k) - \frac{p}{2}\widetilde{C}(k) + o(1) \qquad (\epsilon\to 0)\,.
\end{aligned}
$$

Therefore, using the definition of the constant $C(\Gamma, k)$, we find

$$
\begin{aligned}
\frac{1}{2}\log\left(\det{}'\left(\Delta^1_{\overline{S}_{k+1,\epsilon}}\right)\right) &= \frac{p}{12}\log(\epsilon) + p\left(\frac{k}{2} + \frac{1}{6}\right)\log(-\log(\epsilon)) - \frac{1}{2}\widetilde{C}(\Gamma, k) - \frac{p}{2}\widetilde{C}(k) \\
&\quad + \frac{1}{2}\log\left(\det\nolimits^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)\right) - \frac{p}{6}\log(2) + o(1) \\
&= \frac{p}{12}\log(\epsilon) + p\left(\frac{k}{2} + \frac{1}{6}\right)\log(-\log(\epsilon)) + \frac{1}{2}\log\left(\det\nolimits^*_\Gamma\left(\Delta^1_{\overline{S}_{k+1}}\right)\right) \\
&\quad + C(\Gamma, k) + o(1) \qquad (\epsilon\to 0)\,.
\end{aligned}
$$

Clearing the divergent terms from both sides of formula (4.1.1) and taking the limit for $\epsilon\to 0$ proves the theorem. □

Remark 4.6.2. Theorem 4.6.1 is not the first Riemann–Roch type formula for a modular curve equipped with the hyperbolic metric. In his thesis [24] Freixas proves a theorem of this form, inclusive of constants, for a hermitian line bundle whose complex part is $\overline{M}_1$ on a modular curve with cusps and without elliptic points; the proof combines algebraic deformations on moduli spaces of curves with spectral estimates in Teichmüller theory. In [25] the same author generalizes his result with similar techniques to a hermitian line bundle whose complex part is $\overline{M}_k$ for general k. In his thesis [31] Hahn proves a Riemann–Roch theorem for a hermitian line bundle whose complex part is $\overline{S}_1$ in the presence of cusps and without elliptic points; by comparing the singular objects entering the theorem to their regularized counterparts he is able to prove the theorem up to a topological constant, which he quotes from [24]. Finally, in their recent paper [26] Freixas and von Pippich prove a version of the theorem valid for the trivial sheaf in the presence of both cusps and elliptic points; as a crucial ingredient they use Mayer–Vietoris type formulae to combine determinants of Laplacians obtained via surgery techniques on a complex surface.

4.7 Discussion of the rest terms

The fact that the constant $C(\Gamma, k)$ is implicit is an important limitation of theorem 4.6.1. Since an explicit description of the terms (B) and (C) is necessary for its evaluation, in this section we discuss them. We remark that we expect the divergent part of the asymptotics of the considered terms (B) and (C) to be independent of k, because the divergent term occurring in formula (4.6.1) is independent of k.

Assumption 4.7.1. For simplicity, in this section we restrict to the case $k = 0$.

We begin by explicitly computing the analytic continuation of the terms (B) and (C).

Proposition 4.7.2. *Let γ be the Euler–Mascheroni constant, then the term* (B) *admits the decomposition*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(K^\Gamma_{0,\epsilon}(t;z,z)-K^\Gamma_0(t;z,z)\right)\mu_\epsilon(z),s\right)\right)_{s=0}$$

$$=\int_1^\infty\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(K^\Gamma_{0,\epsilon}(t;z,z)-K^\Gamma_0(t;z,z)-\frac{1}{\mathrm{vol}_\epsilon(X)}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z)\,\frac{\mathrm{d}t}{t} \tag{B1}$$

$$-\gamma\,\mathrm{vol}_{\mathrm{hyp}}\left(X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)\right)\left(\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) \tag{B2}$$

$$+\int_0^1\int_{X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\left(K^\Gamma_{0,\epsilon}(t;z,z)-K^\Gamma_0(t;z,z)\right)\mu_{\mathrm{hyp}}(z)\frac{\mathrm{d}t}{t}. \tag{B3}$$

Proof. The asymptotic expansion for $t\to 0$ of the heat kernel on a compact manifold is given by formulae (45), (46) on page 154 of [13].[1] Denoting by $s_\epsilon(z)$ the scalar curvature at the point z, they state

$$K^\Gamma_{0,\epsilon}(t;z,z) = \frac{1}{4\pi t}+\frac{s_\epsilon(z)}{24\pi}+O_{\Gamma,\epsilon,z}(t)=\frac{1}{4\pi t}-\frac{1}{12\pi}+O_{\Gamma,\epsilon,z}(t) \qquad (t\to 0), \tag{4.7.1}$$

where in the second equality we used the fact that the scalar curvature $s_\epsilon(z)$ equals the hyperbolic scalar curvature at z, i.e., -2, for every $z\in X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)$. Regarding the asymptotic expansion for $t\to 0$ of the hyperbolic heat kernel, by formula (2.2.4) we have

$$K^\Gamma_0(t;z,z)=K_0(t;0)+\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}K_0(t;d_{\mathrm{hyp}}(z,\gamma(z))).$$

Since $X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)$ is compact with a smooth metric, the injectivity radius is uniformly bounded from below

$$\inf_{z\in X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)} i(z)\gg_{\Gamma,\epsilon}1.$$

[1] The value $u_0(x,x)=0$ in the quoted formula (46) is a typo. The correct value $u_0(x,x)=1$ can be deduced by formula (47) of loc. cit..

Thus, by definition of the injectivity radius, there exists a constant $c_{\Gamma,\epsilon}\in\mathbb{R}_{>0}$ such that

$$c_{\Gamma,\epsilon}<\inf_{z\in X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j)}\Bigg(\inf_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}d_{\mathrm{hyp}}(z,\gamma(z))\Bigg).$$

Combining this definition with equation (2.2.5) and the assumption $t\leq 1$, we deduce that

$$\begin{aligned}
&K_0(t;d_{\mathrm{hyp}}(z,\gamma(z)))\\
&\qquad\leq e^{-\frac{c_{\Gamma,\epsilon}^2}{8t}}\frac{\sqrt{2}e^{-\frac{t}{4}}}{(4\pi t)^{\frac{3}{2}}}\int\limits_{d_{\mathrm{hyp}}(z,\gamma(z))}^{\infty}\frac{ue^{-\frac{u^2}{8t}}}{\sqrt{\cosh(u)-\cosh(d_{\mathrm{hyp}}(z,\gamma(z)))}}\,\mathrm{d}u\\
&\qquad\leq\frac{e^{-\frac{c_{\Gamma,\epsilon}^2}{8t}}}{t^{\frac{3}{2}}}e^{\frac{1}{4}}\Bigg(\frac{\sqrt{2}e^{-\frac{1}{4}}}{(4\pi)^{\frac{3}{2}}}\int\limits_{d_{\mathrm{hyp}}(z,\gamma(z))}^{\infty}\frac{ue^{-\frac{u^2}{4}}}{\sqrt{\cosh(u)-\cosh(d_{\mathrm{hyp}}(z,\gamma(z)))}}\,\mathrm{d}u\Bigg)\\
&\qquad=O_\Gamma\Big(e^{-\frac{c_{\Gamma,\epsilon}^2}{9t}}\Big)K_0(1;d_{\mathrm{hyp}}(z,\gamma(z)))\qquad\Bigg(z\in X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j);\,\gamma\neq\mathrm{id};\,t\to0\Bigg).
\end{aligned}$$

Regarding the case $\gamma=\mathrm{id}$, the asymptotic expansion of $K_0(t;0)$ for t small is given by formula (2.6.1). Summing up, we conclude

$$\begin{aligned}
K_0^\Gamma(t;z,z)&=\frac{1}{4\pi t}-\frac{1}{12\pi}+O_{\Gamma,z}(t)+O_\Gamma\Big(e^{-\frac{c_{\Gamma,\epsilon}^2}{9t}}\Big)\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\mathrm{id}}}K_0(1;d_{\mathrm{hyp}}(z,\gamma(z)))\\
&=\frac{1}{4\pi t}-\frac{1}{12\pi}+O_{\Gamma,\epsilon,z}(t)\qquad\Bigg(z\in X\setminus\bigcup_{j=1}^{p}B_\epsilon(P_j);\,t\to0\Bigg).
\end{aligned}$$

Therefore we have

$$K_{0,\epsilon}^\Gamma(t;z,z)-K_0^\Gamma(t;z,z)=O_{\Gamma,\epsilon,z}(t)\qquad(t\to0).\tag{4.7.2}$$

To examine the asymptotic expansion for t large let us recall that $\lambda_{1,\Gamma,\epsilon}$ and $\lambda_{1,\Gamma,\mathrm{hyp}}$ are the first non-zero eingevalues of the Laplacians $\Delta_{0,\epsilon}$ and Δ_0, respectively. By the spectral expansion of the heat kernel on a compact manifold, formula (13) on page 140 of [13], we have

$$K_{0,\epsilon}^\Gamma(t;z,z)=\frac{1}{\mathrm{vol}_\epsilon(X)}+O_{\Gamma,\epsilon,z}\big(e^{-t\lambda_{1,\Gamma,\epsilon}}\big)\qquad(t\to\infty),\tag{4.7.3}$$

combining it with formula (4.5.22) we obtain

$$K_{0,\epsilon}^\Gamma(t;z,z)-K_0^\Gamma(t;z,z)=\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}+O_{\Gamma,\epsilon,z}\big(e^{-t\min\{\lambda_{1,\Gamma,\epsilon},\lambda_{1,\Gamma,\mathrm{hyp}}\}}\big)\qquad(t\to\infty).\tag{4.7.4}$$

Integrating the asymptotic expansions (4.7.2) and (4.7.4) on $X \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)$ we find

$$\int\limits_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \big(K_{0,\epsilon}^{\Gamma}(t;z,z) - K_0^{\Gamma}(t;z,z)\big)\, \mu_{\mathrm{hyp}}(z) = O_{\Gamma,\epsilon}(t) \qquad (t\to 0)\,,$$

and

$$\int\limits_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \big(K_{0,\epsilon}^{\Gamma}(t;z,z) - K_0^{\Gamma}(t;z,z)\big)\, \mu_{\epsilon}(z)$$
$$= \mathrm{vol}_{\mathrm{hyp}}\left(X \setminus \bigcup_{j=1}^{p} B_\epsilon(P_j)\right)\left(\frac{1}{\mathrm{vol}_\epsilon(X)} - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) + O_{\Gamma,\epsilon}\big(e^{-t\min\{\lambda_{1,\Gamma,\epsilon},\lambda_{1,\Gamma,\mathrm{hyp}}\}}\big) \qquad (t\to\infty)\,.$$

Observe that $\mu_\epsilon(z) = \mu_{\mathrm{hyp}}(z)$ for z away from the cusps. We split the domain of integration of the Mellin transform at $t = 1$ and we add and subtract the limit for $t\to\infty$ of the argument, to obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \big(K_{0,\epsilon}^{\Gamma}(t;z,z) - K_0^{\Gamma}(t;z,z)\big)\, \mu_{\epsilon}(z), s\right)\right)_{s=0} \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty \int_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \left(K_{0,\epsilon}^{\Gamma}(t;z,z) - K_0^{\Gamma}(t;z,z)\right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{\mathrm{vol}_\epsilon(X)} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z) t^{s-1}\,\mathrm{d}t\right)_{s=0} \\
&\quad+ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty \int_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \left(\frac{1}{\mathrm{vol}_\epsilon(X)} - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z) t^{s-1}\,\mathrm{d}t\right)_{s=0} \\
&\quad+ \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1 \int_{X\setminus\bigcup_{j=1}^{p} B_\epsilon(P_j)} \left(K_{0,\epsilon}^{\Gamma}(t;z,z) - K_0^{\Gamma}(t;z,z)\right)\mu_{\mathrm{hyp}}(z) t^{s-1}\,\mathrm{d}t\right)_{s=0}.
\end{aligned}$$

The t-integrals appearing in the first and third terms on the right hand side of the last equality are absolutely convergent and holomorphic at $s = 0$ by construction, while the t-integral in the second term has to be intendend in the sense of analytic continuation. We use formula (2.8.2) to deduce

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{t^{s-1}}{\Gamma(s)}\right)_{s=0} = \left(\frac{\log(t)\, t^{s-1}}{\Gamma(s)}\right)_{s=0} - \left(\frac{\big(\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s)\big)\, t^{s-1}}{\Gamma(s)^2}\right)_{s=0} = \frac{1}{t}. \tag{4.7.5}$$

Taking the s-derivative under the t-integral and using this evaluation completes the calculation for the first and third term of the decomposition above. They yield the terms (B1) and (B3), respectively. Since

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty\int_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z)t^{s-1}\,\mathrm{d}t\right)_{s=0}$$
$$=\left(\int_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z)\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\,\mathrm{d}t\right)_{s=0},$$

the relation

$$\int_1^\infty t^{s-1}\mathrm{d}t=-\frac{1}{s}\qquad(\mathrm{Re}(s)<0)\tag{4.7.6}$$

gives the meromorphic continuation of the t-integral in the term under consideration to the whole complex plane. Moreover, the Laurent series of the Γ-function at $s=0$, given by

$$\Gamma(s)=\frac{1}{s}-\gamma+O(s)\qquad(s\to 0)\,,\tag{4.7.7}$$

implies the evaluation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\left(-\frac{1}{s}\right)\right)_{s=0}=\frac{\mathrm{d}}{\mathrm{d}s}\left(-1-\gamma s+O(s^2)\right)_{s=0}=-\gamma.\tag{4.7.8}$$

Therefore, we find

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty\int_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z)t^{s-1}\,\mathrm{d}\,t\right)_{s=0}$$
$$=-\gamma\int\limits_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{\mathrm{vol}_\epsilon(X)}-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z).$$

Since the integrand of the right hand side of the last relation is independent of z, this completes the proof of the proposition. □

Also the term (C) admits a decomposition analogous to the one given for (B).

Proposition 4.7.3. *Let γ be the Euler–Mascheroni constant, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}K^\Gamma_{k,\epsilon}(t;z,z)\,\mu_\epsilon(z),s\right)\right)_{s=0}$$
$$=\int\limits_1^\infty\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(K^\Gamma_{0,\epsilon}(t;z,z)-\frac{1}{\mathrm{vol}_\epsilon(X)}\right)\mu_\epsilon(z)\,\frac{\mathrm{d}t}{t}\tag{C1}$$

$$+\gamma\left(\frac{1}{24\pi}\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z)-\frac{p\,\mathrm{vol}_\epsilon(B_\epsilon(i\infty))}{\mathrm{vol}_\epsilon(X)}\right)-\frac{p\,\mathrm{vol}_\epsilon(B_\epsilon(i\infty))}{4\pi} \tag{C2}$$

$$+\int\limits_0^1\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(K^\Gamma_{0,\epsilon}(t;z,z)-\frac{1}{4\pi t}-\frac{s_\epsilon(z)}{24\pi}\right)\mu_\epsilon(z)\,\frac{\mathrm{d}t}{t}. \tag{C3}$$

Proof. The asymptotic expansions of $K^\Gamma_{0,\epsilon}(t;z,z)$ for t small and large are given by formulae (4.7.1) and (4.7.3), respectively. Splitting the domain of integration of the Mellin trasnform at $t=1$ and adding and subtracting the constant and diverging parts of the asymptotic expansions for t small and large, we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\mathcal{M}\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} K^\Gamma_{k,\epsilon}(t;z,z)\,\mu_\epsilon(z),s\right)\right)_{s=0}\\
&\quad=\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(K^\Gamma_{k,\epsilon}(t;z,z)-\frac{1}{\mathrm{vol}_\epsilon(X)}\right)\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\frac{1}{\mathrm{vol}_\epsilon(X)}\,\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(K^\Gamma_{k,\epsilon}(t;z,z)-\frac{1}{4\pi t}-\frac{s_\epsilon(z)}{24\pi}\right)\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}\\
&\qquad+\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{4\pi t}+\frac{s_\epsilon(z)}{24\pi}\right)\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}.
\end{aligned}$$

The t-integrals in the first and third terms of the last expression are holomorphic at $s=0$ by construction, and the contribution of the corresponding terms is computed via formula (4.7.5) as in the proof of the previous proposition. They yield the terms (C1) and (C3), respectively. The contribution of the second term in the last decomposition is computed with the use of formulae (4.7.6) and (4.7.8) as in the previous proposition, and, noting that $\mathrm{vol}_\epsilon(B_\epsilon(P_j))=\mathrm{vol}_\epsilon(B_\epsilon(i\infty))$ for any cusp P_j, it has the value

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_1^\infty\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\frac{1}{\mathrm{vol}_\epsilon(X)}\,\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}=-\gamma\,\frac{p\,\mathrm{vol}_\epsilon(B_\epsilon(i\infty))}{\mathrm{vol}_\epsilon(X)}.$$

It remains to establish the contribution of the fourth term of the decomposition. Since

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\left(\frac{1}{4\pi t}+\frac{s_\epsilon(z)}{24\pi}\right)\mu_\epsilon(z)\,t^{s-1}\mathrm{d}t\right)_{s=0}\\
&\quad=\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\frac{1}{4\pi}\mu_\epsilon(z)\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-2}\mathrm{d}t\right)_{s=0}\\
&\qquad+\left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)}\frac{s_\epsilon(z)}{24\pi}\mu_\epsilon(z)\right)\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\mathrm{d}t\right)_{s=0},
\end{aligned}$$

the relations

$$\int_0^1 t^{s-2}\mathrm{d}t = \frac{1}{s-1} \qquad (\mathrm{Re}(s) > 1)$$

and

$$\int_0^1 t^{s-1}\mathrm{d}t = \frac{1}{s} \qquad (\mathrm{Re}(s) > 0)$$

give the meromorphic continuation of the t-integral in the term under consideration to the whole complex plane. Using the Laurent expansion for the Γ-function at $s = 0$, formula (4.7.7), we deduce the values

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\frac{1}{s}\right)_{s=0} = \gamma,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\frac{1}{s-1}\right)_{s=0} = -1.$$

Thus, the fourth term has the value

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{\Gamma(s)}\int_0^1 \int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(\frac{1}{4\pi t} + \frac{s_\epsilon(z)}{24\pi}\right)\mu_\epsilon(z)\, t^{s-1}\mathrm{d}t\right)_{s=0}$$
$$= -\frac{p\,\mathrm{vol}_\epsilon(B_\epsilon(i\infty))}{4\pi} + \frac{\gamma}{24\pi}\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z).$$

This completes the proof of the proposition. □

We now provide partial results for the asymptotic expansions of the terms appearing in propositions (4.7.2) and (4.7.3). The term (B2) has the limit

$$\lim_{\epsilon\to 0}\gamma\,\mathrm{vol}_{\mathrm{hyp}}\left(X \setminus \bigcup_{j=1}^p B_\epsilon(P_j)\right)\left(-\frac{1}{\mathrm{vol}_\epsilon(X)} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) = 0,$$

because $\mathrm{vol}_\epsilon(X)$ converges to $\mathrm{vol}_{\mathrm{hyp}}(X)$ for $\epsilon\to 0$ by construction. We use the Gauss–Bonnet theorem to show that also the term (C2) has a finite limit for $\epsilon\to 0$.

Proposition 4.7.4. *The following asymptotic expansion holds*

$$\gamma\left(\frac{1}{24\pi}\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z) - \sum_{j=1}^p \frac{\mathrm{vol}_\epsilon(B_\epsilon(P_j))}{\mathrm{vol}_\epsilon(X)}\right) - \sum_{j=1}^p \frac{\mathrm{vol}_\epsilon(B_\epsilon(P_j))}{4\pi}$$
$$= \frac{\gamma\, p}{6} + o(1) \qquad (\epsilon\to 0)\,.$$

Proof. Since $\lim\limits_{\epsilon\to 0} \mathrm{vol}_\epsilon(B_\epsilon(i\infty)) = 0$, we only need to examine the integral of the scalar curvature, which is twice the Gaussian curvature $K_\epsilon(z)$ because we are working in real dimension 2. We obtain

$$\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z) = 2 \int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} K_\epsilon(z)\,\mu_\epsilon(z).$$

By the Gauss–Bonnet theorem, the integral of the Gaussian curvature on a compact surface is 2π times its Euler characteristic. Therefore we deduce

$$\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z) = 4\pi\,\chi(X) - 2 \int\limits_{X\setminus\bigcup_{j=1}^p B_\epsilon(P_j)} K_\epsilon(z)\,\mu_\epsilon(z).$$

The Euler characteristic of X has the value $\chi(X) = 2-2g$. Moreover $z \in X \setminus \bigcup_{j=1}^p B_\epsilon(P_j)$ implies $K_\epsilon(z) = K_{\mathrm{hyp}}(z) = -1$ and $\mu_\epsilon(z) = \mu_{\mathrm{hyp}}(z)$. Thus

$$\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z) = 8\pi(1-g) + 2\,\mathrm{vol}_{\mathrm{hyp}}\left(X \setminus \bigcup_{j=1}^p B_\epsilon(P_j)\right).$$

Since $\mathrm{vol}_{\mathrm{hyp}}(X) = 2\pi(2g-2+p)$ and $\lim\limits_{\epsilon\to 0} \mathrm{vol}_{\mathrm{hyp}}(B_\epsilon(i\infty)) = 0$, we conclude

$$\int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} s_\epsilon(z)\,\mu_\epsilon(z) = 4\pi\,p + o(1) \qquad (\epsilon\to 0)\,.$$

This completes the proof of the proposition. □

We are going to show that also the term (C1) goes to zero for $\epsilon\to 0$. We begin by providing a positive lower bound for the first non-zero eigenvalue $\lambda_{1,\Gamma,\epsilon}$ of $\Delta_{0,\epsilon}$ which is uniformly valid through metric degeneration, i.e., in the limit for $\epsilon\to 0$. To prove it, we first introduce the notion of Cheeger constant of a Riemannian manifold.

Definition 4.7.5. Let M be a Riemannian manifold, and let $\alpha \subset M$ be the smooth boundary of a submanifold $A_\alpha \subset M$ with smaller volume than its complementary. The *Cheeger value of* α is the positive real number

$$h(\alpha) := \frac{\ell(\alpha)}{\mathrm{vol}(A_\alpha)}.$$

The *Cheeger constant of* M is the positive real number

$$h(M) := \inf_{\alpha\subset M} h(\alpha),$$

where the infimum is taken over the smooth boundaries $\alpha \subset M$. Finally, we say that the smooth boundary α is a *solution to the Cheeger isoperimetric problem on* M if

$$h(\alpha) = h(M).$$

We now give similar definitions for closed Riemannian manifolds with boundary. Observing that, since the presence of a boundary implies a minimal perimeter for large volumes, the Cheeger isoperimetric problem is well-posed even without requiring that A_α has smaller volume than its complementary.

Definition 4.7.6. Let M be a closed Riemannian manifold with smooth boundary, and $\alpha \subset M$ a smooth, closed and non self-intersecting curve with interior A_α. The *Cheeger value with boundary of* α is the positive real number

$$h_b(\alpha) := \frac{\ell(\alpha)}{\mathrm{vol}(A_\alpha)}.$$

The *Cheeger constant with boundary of* M is the positive real number

$$h_b(M) := \inf_{\alpha \subset M} h_b(\alpha),$$

where the infimum is taken over the smooth, closed and non self-intersecting curves $\alpha \subset M$. Finally, we say that the smooth, closed non self-intersecting curve α is a *solution to the Cheeger isoperimetric problem with boundary on* M if

$$h_b(\alpha) = h_b(M).$$

Observe that if we restrict to $\alpha \subset N \subset M$, where N is a submanifold of M with smooth boundary and volume smaller than its complementary, then $h_b(\alpha)$ taken with respect to N equals $h(\alpha)$ taken with respect to M. Our interest in the Cheeger constant comes from an inequality proved by Cheeger in [14], which states

$$\lambda_{1,M} \geq \frac{h\,(M)^2}{4}, \tag{4.7.9}$$

where $\lambda_{1,M}$ is the first non-zero eigenvalue of the Laplace–Beltrami operator of M. We now characterize solutions of the Cheeger isoperimetric problem with boundary on a closed disk with radially symmetric and monotone non-increasing metric.

Lemma 4.7.7. *Let D be a closed disk with radially symmetric and monotone non-increasing metric, i.e., with volume form $\frac{dr\wedge d\theta}{\rho(r)^2}$ with $\rho(r)$ monotone non-decreasing. Then, the solution to the Cheeger isoperimetric problem with boundary on D is the boundary ∂D of the disk itself.*

Proof. The proof of the lemma is articulated in three steps: First we prove the existence of a solution, then we prove that the solution must contain the center of the disk, and finally that the solution must be a circle. If not otherwise specified, in this proof we use the word curve for a smooth, closed non self-intersecting curve $\alpha \subset D$, and we denote by A_α its interior. Moreover, we do not specify anymore that we consider the version of the Cheeger isoperimetric problem with a boundary. To prove that a solution exists, we observe that the function defined by the Cheeger value with boundary

$$h_b\colon \{\alpha \subset D \text{ smooth, closed, non self-intersecting}\} \longrightarrow \mathbb{R}_{>0}$$

factorizes through the map

$$\varphi\colon \{\alpha \subset D \text{ smooth, closed, non self-intersecting}\} \longrightarrow \mathbb{R}^2_{>0},$$

given by the assignment

$$\varphi(\alpha) = (\ell(\alpha), \mathrm{vol}(A_\alpha))\,,$$

and the elementary map $(u, v) \mapsto \frac{u}{v}$. We provide bounds on the candidates to be solutions to the Cheeger isoperimetric problem. A curve $\alpha \subset D$ with $\ell(\alpha) > \ell(\partial D)$ has Cheeger value strictly larger than ∂D. Also curves with too small length cannot be solutions to the Cheeger isoperimetric problem: Theorem 2.2 of [38] states that minimal perimeter solutions for small volume are spheres, whose Cheeger values blow up for small perimeter. We can therefore restrict to curves α with $\ell(\partial D) \geq \ell(\alpha) \geq c_{D,\rho} \in \mathbb{R}_{>0}$. We can argue in a similar way for the volume: For each curve α it holds $\mathrm{vol}(A_\alpha) \leq \mathrm{vol}(D)$, and, by the above-mentioned theorem, we can restrict to $\mathrm{vol}(A_\alpha) \geq d_{D,\rho} \in \mathbb{R}_{>0}$. In particular we are reduced to prove that the function $(u, v) \mapsto \frac{u}{v}$ achieves a minimum on

$$\mathrm{Im}(\varphi) \cap \{(u, v) \,|\, c_{D,\rho} \leq u \leq \partial D,\, d_{D,\rho} \leq v \leq \mathrm{vol}(D)\}\,.$$

The function $(u, v) \mapsto \frac{u}{v}$ is continuous away from $v = 0$, and the domain is bounded. Thus it remains to prove that it is also closed, thus compact. Since $\mathrm{vol}(A_\alpha) < \mathrm{vol}(D)$ implies that for any $l > \ell(\alpha)$ we can find a curve β with $\ell(\beta) = l$ and $\mathrm{vol}(A_\beta) = \mathrm{vol}(A_\alpha)$, it is enough to show that the isovolumetric problem of volume V has a minimal perimeter solution for each $V \in [d_{D,\rho}, \mathrm{vol}(D)]$ fixed. For $V \in [d_{D,\rho}, \mathrm{vol}(D))$ this is proven in theorem 3.4 of [32], while for $V = \mathrm{vol}(D)$ the solution is obviously ∂D. This proves that a solution to this Cheeger problem with boundary exists.

Let α be a solution to the Cheeger problem, whose existence we just proved. We now show that the origin must lie inside α proceeding by contradiction, i.e., we assume $0 \notin A_\alpha$. In the argument we use the fact that the space of smooth curves in the disk is dense in the space of continuous curves in the disk with respect to the Hausdorff metric, and the Cheeger function h_b is continuous with respect to this metric. We can trivially exclude the cases where α has more than one connected component. We first show that there are exactly two distinct semi-lines extending from 0 that are tangent to α. Since α is connected, there cannot be more than two. Assume that there are none, then there must be a semi-line l extending from 0 and properly intersecting α in at least 4 points. Let p and q be the second and third point of intersection of l and α, respectively. Let $\alpha_{p,q}$ and $l_{p,q}$ be the arc on α connecting p and q and containing the first intersection point of α and l and the arc of l connecting p and q, respectively. Since the metric on the disk is radially symmetric, the shortest path connecting p and q is $l_{p,q}$. Therefore, replacing the arc $\alpha_{p,q}$ by $l_{p,q}$ on α we obtain a continuous curve with shorter lenght and larger volume than α, i.e., with a strictly smaller Cheeger value. It is therefore possible to find a smooth curve with a Cheeger value strictly smaller than α. Contradiction. Similarly, if there is only one semi-line tangent to α it must have at least two points of tangency p and q. Using the same argument we derive a contradiction.
Let l_0 be a semi-line starting from 0 and tangent to α, and define θ_0 to be the smallest angle such that α is contained in the cone limited by l_0 and its θ_0-rotation. We denote

this second semi-line by l_1, which, by minimality of θ_0, is the second tangent from 0 to α. If $\theta_0 \leq \pi$ we replace α by α_1, which is given by α plus its reflection through the semi-line l_1. Observe that $h_b(\alpha) = h_b(\alpha_1)$. If $\theta_1 = 2\theta_0 \leq \pi$ we repeat the same construction: The tangents to α_1 are l_0 and its θ_1-rotation l_2, then we replace α_1 by α_2, given by the union of α_1 and its reflection through l_2. We repeat this construction until $2\pi \geq \theta_n > \pi$. The curve α_n is still a solution to the Cheeger isoperimetric problem, because this construction does not change the Cheeger value. Since the metric is radially symmetric, and $\theta_n > \pi$, the geodesic connecting the two extremal points of α_n, i.e., the first tangency point of l_0 and the first tangency point of l_{n+1}, is strictly shorter than the shortest path connecting them along α_n. The curve obtained replacing in α the shortest arc connecting the extremal points by the geodesic connecting them has also larger inscribed area than α, thus a smaller Cheeger value. This is the desired contradiction that proves $0 \in A_\alpha$.

We finally prove that the solution must be a circle centered in 0. This argument already appeared in [36]. Let l be a line passing through 0 and splitting α into two, not necessarily connected, parts. By abuse of notation we call the Cheeger value of one of the two sides of α the ratio of the length of the part of α on that side of l and the area of A_α on that side of l. If the two sides did not have the same Cheeger value, we could replace α by the side with the smallest Cheeger value plus its reflection through l. This curve would only be continuous, but by density we could find a smooth curve with Cheeger value arbitrarily close to its Cheeger value. This contradicts the fact that α is a solution. Therefore, every line through the origin splits α in two parts with Cheeger values equal to $h_b(\alpha)$. If α is not a circle, there exists a line n through the origin that intersects it not orthogonally. Replacing α by one of the halves in the splitting by n plus its reflection does not vary the Cheeger value, but yields a curve with a concavity directed away from the origin. Since the metric is monotone non-increasing, the convex hull of this curve has smaller perimeter and larger area than α, thus a smaller Cheeger value. This gives the desired contradiction.

To complete the proof of the lemma it remains to show that the boundary ∂D has a smaller Cheeger value than all the other circles centered in the origin. This follows immediately from the fact that the metric on the disk is monotone non-increasing. $\square$

Theorem 10.1 of [36] states that the Cheeger isoperimetric problem on a hyperbolic surface with at least one cusp has a solution given by a union of geodesics or a horocycle around a cusp. In particular, denoting by $h\left(Y(\Gamma)_{\text{hyp}}\right)$ the Cheeger constant of $Y(\Gamma)$ equipped with the hyperbolic metric, there exists a smooth boundary $\alpha_\Gamma \subset Y(\Gamma) \subset X$ such that

$$h\left(Y(\Gamma)_{\text{hyp}}\right) = h(\alpha_\Gamma).$$

Recall that we fixed $s_\Gamma \in \mathbb{R}_{>0}$ such that $B_{s_\Gamma}(P_j) \cap B_{s_\Gamma}(P_h) = \emptyset$ for $j \neq h$ and $-\log(s_\Gamma) > 2\pi$.

Lemma 4.7.8. *Let $\lambda_{1,\Gamma,\epsilon}$ be the first non-zero eigenvalue of $\Delta_{0,\epsilon}$, then there exists $\delta_\Gamma \in \mathbb{R}_{>0}$ such that for $0 < \epsilon < \delta_\Gamma$ we have the inequality*

$$\lambda_{1,\Gamma,\epsilon} \geq \frac{h\left(Y(\Gamma)_{\text{hyp}}\right)^2}{4}.$$

Proof. Write $h\left(X_\epsilon\right)$ for the Cheeger constant of the surface X equipped with the ϵ-regularized hyperbolic metric. Then the lemma is implied by Cheeger inequality (4.7.9) and by the relation

$$h\left(X_\epsilon\right) \geq h\left(Y(\Gamma)_{\mathrm{hyp}}\right) \tag{4.7.10}$$

for ϵ small enough, which we now prove. For a smooth boundary $\alpha \subset Y(\Gamma) \subset X$ we write

$$h_\epsilon(\alpha) := \frac{\ell_\epsilon(\alpha)}{\mathrm{vol}_\epsilon(A_\alpha)}$$

and

$$h_{\mathrm{hyp}}(\alpha) := \frac{\ell_{\mathrm{hyp}}(\alpha)}{\mathrm{vol}_{\mathrm{hyp}}(A_\alpha)}$$

for the Cheeger value of α with respect to the ϵ-regularized metric and the hyperbolic metric, respectively. Without loss of generality we restrict to ϵ small enough such that $B_\epsilon(P_j) \cap \alpha_\Gamma = \emptyset$ for each cusp P_j. We compute

$$\mathrm{vol}_\epsilon(B_\epsilon(i\infty)) = \int\limits_{B_\epsilon(i\infty)} \frac{r\,\mathrm{d}r \wedge \mathrm{d}\theta}{4\pi^2 \rho_{1,\epsilon}(r)^2} \leq \int\limits_0^{2\pi}\int\limits_0^{\epsilon} \frac{r\,\mathrm{d}r \wedge \mathrm{d}\theta}{\left(\phi(\epsilon)\log\phi(\epsilon)\right)^2} = \frac{2\pi\epsilon^2}{2\left(\phi(\epsilon)\log\phi(\epsilon)\right)^2} \leq \frac{2\pi}{\log(\epsilon)^2}, \tag{4.7.11}$$

where in the last inequality we used $\epsilon > \phi(\epsilon) \geq \frac{\epsilon}{2}$, valid for ϵ small enough. Combining this estimate with

$$\mathrm{vol}_{\mathrm{hyp}}(B_\epsilon(i\infty)) = -\frac{2\pi}{\log(\epsilon)},$$

we obtain the inequality $\mathrm{vol}_\epsilon(B_\epsilon(P_j)) < \mathrm{vol}_{\mathrm{hyp}}(B_\epsilon(P_j))$ for ϵ small enough, which implies $h_\epsilon(\alpha_\Gamma) \geq h_{\mathrm{hyp}}(\alpha_\Gamma)$. We assume, in contradiction to equation (4.7.10), the existence of a smooth boundary $\alpha \subset X$ such that

$$h_\epsilon(\alpha) < h_{\mathrm{hyp}}(\alpha_\Gamma).$$

Then, one of the following three non-disjoint cases must occur:

(1) For every cusp P_j it holds $\alpha \cap B_\epsilon(P_j) = \emptyset$.

(2) There exists a cusp P_j such that $\alpha \cap B_\epsilon(P_j) \neq \emptyset$, but $\alpha \not\subset B_{s_\Gamma}(P_j)$.

(3) There exists a cusp P_j such that $\alpha \subset B_{s_\Gamma}(P_j)$.

In the first case, since $\ell_\epsilon(\alpha) = \ell_{\mathrm{hyp}}(\alpha)$, $\mathrm{vol}_\epsilon(B_\epsilon(P_j)) < \mathrm{vol}_{\mathrm{hyp}}(B_\epsilon(P_j))$ and α_Γ is a solution to the Cheeger isoperimetric problem on $Y(\Gamma)$ with the hyperbolic metric, we find

$$h_\epsilon(\alpha) \geq h_{\mathrm{hyp}}(\alpha) \geq h_{\mathrm{hyp}}(\alpha_\Gamma),$$

which is a contradiction. In the second case the length of α has to be at least twice $d_\epsilon(\partial B_{s_\Gamma}(P_j), \partial B_\epsilon(P_j))$, and the volume of A_α can be at most one half of the total volume of X. Thus, we find

$$\begin{aligned} h_\epsilon(\alpha) =& \frac{\ell_\epsilon(\alpha)}{\mathrm{vol}_\epsilon(A_\alpha)} \geq \frac{4d_\epsilon(\partial B_{s_\Gamma}(P_j), \partial B_\epsilon(P_j))}{\mathrm{vol}_\epsilon(X)} = \frac{4d_{\mathrm{hyp}}(\partial B_{s_\Gamma}(P_j), \partial B_\epsilon(P_j))}{\mathrm{vol}_\epsilon(X)} \\ \geq& \frac{4d_{\mathrm{hyp}}(\partial B_{s_\Gamma}(P_j), \partial B_\epsilon(P_j))}{\mathrm{vol}_{\mathrm{hyp}}(X)} = \frac{4d_{\mathrm{hyp}}\left(-i\frac{\log(s_\Gamma)}{2\pi}, -i\frac{\log(\epsilon)}{2\pi}\right)}{\mathrm{vol}_{\mathrm{hyp}}(X)} = \frac{4\log\left(\frac{\log(\epsilon)}{\log(s_\Gamma)}\right)}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \end{aligned}$$

The latter quantity can be made arbitrarily large by choosing ϵ small enough, therefore also this case leads to a contradiction. For the third case we observe that, without loss of generality, the volume of the cusp is less than half of the total volume of the manifold. Moreover, the closure of the ball $B_{s_\Gamma}(P_j)$ is a disk equipped with a radially symmetric monotone non-decreasing metric by construction. Therefore lemma 4.7.7 implies

$$h_\epsilon(\alpha) \geq h_\epsilon(\partial B_{s_\Gamma}(P_j)) = h_{\mathrm{hyp}}(\partial B_{s_\Gamma}(P_j)) \geq h_{\mathrm{hyp}}(\alpha_\Gamma).$$

This is a contradiction to the third and last case, and proves therefore the statement of the lemma. □

We now estimate the on-diagonal heat kernel $K^\Gamma_{0,\epsilon}(t; z, z)$ for z close to a cusp and $t = 1$.

Lemma 4.7.9. *Let $z \in B_\epsilon(i\infty)$ and $t = 1$, then we have the estimate*

$$K^\Gamma_{0,\epsilon}(1; z, z) \ll -\log(\epsilon).$$

Proof. Li and Yau proved in [43, Corollary 3.1] that, for any $z \in X$, the heat kernel is bounded by

$$K^\Gamma_{0,\epsilon}(t; z, z) \ll_{\dim(X)} \frac{e^{c_{\dim(X)}K(\epsilon)t}}{\mathrm{vol}_\epsilon\left(B^\epsilon_{\sqrt{t}}(z)\right)},$$

where $c_{\dim(X)}$ is a universal constant only depending on the dimension of X, $B^\epsilon_{\sqrt{t}}(z)$ is the geodesic ball of center z and radius $\sqrt{t}$ associated to the metric μ_ϵ, and $K(\epsilon)$ is a constant such that the Ricci curvature admits the bound $\mathrm{Ric}_\epsilon(z) \geq -K(\epsilon)$ uniformly on X. Since X is a compact surface, we can choose

$$-K(\epsilon) = \min_{z\in X} K_\epsilon(z),$$

where $K_\epsilon(z)$ is the Gaussian curvature at z. The Gaussian curvature is -1 for $z \in X \setminus \bigcup_{j=1}^p B_\epsilon(P_j)$, it is large and positive on the regularization annuli $B_\epsilon(P_j) \setminus B_{\phi(\epsilon)}(P_j)$ and it is zero on the neighborhoods $B_{\phi(\epsilon)}(P_j)$. Setting $t = 1$, we are left to prove

$$\mathrm{vol}_\epsilon\left(B^\epsilon_1(z)\right) \gg \frac{1}{\log(\epsilon)}, \tag{4.7.12}$$

for $z \in B_\epsilon(P_j)$. Without loss of generality, we assume $z \in B_\epsilon(i\infty)$. Let $\mathrm{diam}_\epsilon\left(B_\epsilon(i\infty)\right)$ be the diameter of $B_\epsilon(i\infty)$ calculated according to the ϵ-regularized metric. Writing $\mathrm{diam}_{\mathrm{Euc}}$ for the euclidean diameter and using the inequality $\epsilon > \phi(\epsilon) \geq \frac{\epsilon}{2}$, we find

$$\mathrm{diam}_\epsilon\left(B_\epsilon(i\infty)\right) \leq \frac{\mathrm{diam}_{\mathrm{Euc}}\left(B_\epsilon(i\infty)\right)}{-\phi(\epsilon)\log\phi(\epsilon)} = \frac{2\epsilon}{-\phi(\epsilon)\log\phi(\epsilon)} \leq -\frac{4}{\log(\epsilon)}.$$

Therefore $\mathrm{diam}_\epsilon\left(B_\epsilon(i\infty)\right)$ is decreasing in ϵ and we can choose ϵ smal enough such that $\mathrm{diam}_\epsilon\left(B_\epsilon(i\infty)\right) \leq \frac{1}{2}$. For any $z \in B_\epsilon(i\infty)$, we obtain the inclusion

$$B_\epsilon(i\infty) \cup \left\{ w \in X \setminus B_\epsilon(i\infty) \,\middle|\, d_\epsilon(w, \partial B_\epsilon(i\infty)) \leq \frac{1}{2} \right\} \subseteq B_1^\epsilon(z).$$

We deduce that the quantity $\mathrm{vol}_\epsilon\left(B_1^\epsilon(z)\right)$ is bounded from below by the ϵ-volume of the left hand side, which we now compute. Since it is positive we ignore the contribution of $\mathrm{vol}_\epsilon(B_\epsilon(i\infty))$. Regarding the other term, the computation

$$d_\epsilon\left(w, \partial B_\epsilon(i\infty)\right) = d_{\mathrm{hyp}}\left(i\mathrm{Im}(w), -\frac{i\log(\epsilon)}{2\pi}\right) = \log\left(-\frac{\log(\epsilon)}{2\pi\mathrm{Im}(w)}\right),$$

implies the equality

$$\left\{ w \in X \setminus B_\epsilon(i\infty) \,\middle|\, d_\epsilon(w, \partial B_\epsilon(i\infty)) \leq \frac{1}{2} \right\} = \mathcal{F}_\infty\left(-\frac{\log(\epsilon)}{2\pi\sqrt{e}} < y < -\frac{\log(\epsilon)}{2\pi}\right).$$

Therefore we have

$$\mathrm{vol}_\epsilon\left(\left\{ w \in X \setminus B_\epsilon(i\infty) \,\middle|\, d_\epsilon(w, \partial B_\epsilon(i\infty)) \leq \frac{1}{2} \right\}\right) = \int\limits_{-\frac{\log(\epsilon)}{2\pi\sqrt{e}}}^{-\frac{\log(\epsilon)}{2\pi}} \frac{\mathrm{d}y}{y^2} = -\frac{2\pi\left(\sqrt{e}-1\right)}{\log(\epsilon)}.$$

This proves the desired lower bound (4.7.12). and completes the proof of the lemma. $\square$

Now we use lemmata 4.7.8 and (4.7.9) to estimate the term (C1).

Proposition 4.7.10. *The following estimate holds*

$$\int\limits_1^\infty \int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(t;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)}\right)\mu_\epsilon(z)\,\frac{\mathrm{d}t}{t} \ll_\Gamma -\frac{1}{\log(\epsilon)}.$$

Proof. By the spectral expansion of the heat kernel on a compact manifold, formula (13) on page 140 of [13], we have

$$\int\limits_1^\infty \int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(t;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)}\right)\mu_\epsilon(z)\,\frac{\mathrm{d}t}{t}$$

$$= \int\limits_1^\infty \int\limits_{\bigcup_{j=1}^p B_\epsilon(P_j)} \sum_{l\geq 1} e^{-\lambda_{l,\Gamma,\epsilon}t}\left|\varphi_{l,\epsilon}(z)\right|^2 \mu_\epsilon(z)\,\frac{\mathrm{d}t}{t}.$$

Let h_Γ be the positive uniform lower bound given by lemma 4.7.8. For $l \geq 1$ and for ϵ small enough it holds $h_\Gamma < \lambda_{l,\Gamma,\epsilon}$. Further applying Tonelli's theorem, we find

$$\int_1^\infty \int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(t;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)} \right) \mu_\epsilon(z) \frac{\mathrm{d}t}{t}$$
$$\leq \left(\int_1^\infty \frac{e^{-h_\Gamma t}}{t} \mathrm{d}t \right) \left(\int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \sum_{l \geq 1} e^{-\left(\lambda_{l,\Gamma,\epsilon} - h_\Gamma\right)} \left|\varphi_{l,\epsilon}(z)\right|^2 \mu_\epsilon(z) \right).$$

The first factor is a universal constant depending on Γ, while the integrand in the second one can be rewritten as a heat kernel evaluated at $t = 1$. We obtain

$$\int_1^\infty \int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(t;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)} \right) \mu_\epsilon(z) \frac{\mathrm{d}t}{t}$$
$$\ll_\Gamma e^{h_\Gamma} \int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(1;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)} \right) \mu_\epsilon(z).$$

By lemma 4.7.9, using the fact that scaling matrices are isometries between cusp neighborhoods, the integrand is bounded by a universal constant times $-\log(\epsilon)$. Moreover, by formula (4.7.11), the volume of the domain of integration is bounded by $p\, \frac{2\pi}{\log(\epsilon)^2}$. Therefore

$$\int_1^\infty \int_{\bigcup_{j=1}^p B_\epsilon(P_j)} \left(K_{0,\epsilon}^\Gamma(t;z,z) - \frac{1}{\mathrm{vol}_\epsilon(X)} \right) \mu_\epsilon(z) \frac{\mathrm{d}t}{t} \ll_\Gamma -\frac{1}{\log(\epsilon)}.$$

This completes the proof of the proposition. □

Appendix A

An eigenfunction expansion associated to the Whittaker equation

In remark 3.4.2 we observed that theorem 3.4.1 can be seen as a special case of a generalization of a heat kernel formula of Müller [46, formula (2.29)]; where the generalization consists in considering the heat kernel associated to the Laplacian on $(k,0)$-tensors instead of the Laplacian on functions, and the specialization consists in fixing the base manifold to S^1 and in restricting to the on-diagonal heat kernel. A crucial spectral tool for this result is formula (1.14) of Müller [46], originally proven by Lebedev [42] as an inversion formula for the Kontorovich–Lebedev transform, which we now describe. Let $K_{ir}(u)$ be the modified Bessel function of the second kind with purely imaginary parameter. The Kontorovich–Lebedev transform of the test function $g \in \mathcal{C}_0^\infty(\mathbb{R}_{>0})$ is given by

$$\int_0^\infty K_{ir}(v)g(v)\frac{\mathrm{d}v}{v},$$

and it admits the explicit inversion formula

$$g(u) = \frac{2}{\pi^2}\int_0^\infty r\sinh(\pi r)K_{ir}(u)\int_0^\infty K_{ir}(v)g(v)\frac{\mathrm{d}v}{v}\,\mathrm{d}r.$$

This inversion formula can be proven by Weyl–Titchmarsh theory as an eigenfunction expansion associated to the modified Bessel equation in Sturm–Liouville form, see example 4.15 of [63]. In the last seventy years this result has been generalized in a number of ways, we refer to [68] for an overview on this topic. Recently, partly due to the many applications in theoretical physics, there has been an increased interest in integral transforms with kernel a Whittaker functions with purely imaginary second parameter [60, 67, 2, 6, 61]. Thus, one may want to generalize this result substituting the modified Bessel function of the second kind $K_{ir}(u)$ by the Whittaker function $W_{\kappa,ir}(u)$, since for $\kappa = 0$ it holds the elementary relation

$$W_{0,ir}(2u) = \sqrt{\frac{2u}{\pi}}K_{ir}\left(u\right).$$

In practice this generalization can be approached in two different ways. On one hand it is possible to define an integral transform with a Whittaker function as kernel; this has been done in [60], where the authors study the integral transform

$$\int_{-\infty}^\infty r\sinh(2\pi r)\Gamma\left(-\kappa+\frac{1}{2}-ir\right)\Gamma\left(-\kappa+\frac{1}{2}+ir\right)W_{\kappa,ir}(u)g(r)\,\mathrm{d}r \qquad (\kappa\in\mathbb{C};\ u\in\mathbb{R}),$$

which directly generalizes the inverse Kontorovich–Lebedev transform. On the other hand, it is possible to write the Whittaker equation with parameters κ and ir in Sturm–Liouville form and write an expansion in terms of its eigenfunctions. In general, these two approaches do not coincide, because of the presence of a discrete spectrum if $\mathrm{Re}(\kappa) \geq 1$. The purpose of this section is to realize the latter idea in the case $\kappa \in \mathbb{Z}$. We remark that a similar approach has been studied in [2] under different constraints, the important novelty of our result is that it allows the presence of the discrete spectrum.
We first state a preliminary lemma about possible values of generalized Laguerre polynomials evaluated at 1.

Lemma A.1. *Let $k \geq 2$ and $j \in \{0, ..., k-1\}$, then*

$$L_j^{(2k-2j-1)}(1) \neq 0, \tag{A.1}$$

and

$$k\, L_j^{(2k-2j-1)}(1) \neq (j+1)\, L_{j+1}^{(2k-2j-1)}(1), \tag{A.2}$$

where $L_n^{(\alpha)}(Z)$ is the n-th generalized Laguerre polynomial with parameter α.

Proof. Observe that the finite power series formula for Laguerre polynomials, given formula 18.5.12 of [48] and which states

$$L_n^{(\alpha)}(Z) = \sum_{l=0}^{n} \frac{(\alpha+l+1)_{n-l}}{(n-l)!\, l!}(-Z)^l, \tag{A.3}$$

implies

$$n!\, L_n^{(m)}(1) = \sum_{l=0}^{n} \frac{n!\,(n+m)!}{m!\,(n-l)!\,l!}(-1)^l \equiv \sum_{l=a}^{n} \binom{n+m}{n-l} \frac{n!\,(-1)^l}{l!} \pmod{a}$$
$$(n, m \in \mathbb{N}, a \in \mathbb{N}_{\leq n})\,. \tag{A.4}$$

If $j \geq 2$, choosing $a = n = j$ and $m = 2k-2j-1$ in the formula above proves equation (A.1). While, if $j = 0, 1$, the statement is proven by the explicit expressions

$$L_0^{(m)}(1) = 1, \qquad L_1^{(m)}(1) = m.$$

We now prove the second assertion. Applying equation (A.4) with $n = a = j$ and $m = 2k-2j-1$, and with $n = j+1$, $a = j$ and $m = 2k-2j-1$ gives

$$k\, j!\, L_j^{(2k-2j-1)}(1) \equiv k\,(-1)^j \pmod{j},$$

and

$$(j+1)!\, L_{j+1}^{(2k-2j-1)}(1) \equiv (-1)^{j+1} + (-1)^j(2k-j)(j+1) \pmod{j},$$

respectively. Let us assume by contradiction that we have an equality in equation (A.2), then

$$k(-1)^j \equiv (-1)^{j+1} + (-1)^j (2k-j)(j+1) \pmod{j},$$

i.e.,

$$k \equiv 1 \pmod{j}.$$

We want to prove that the last equivalence implies $k = j+1$. We make a second assumption by contradiction, namely that the equality $k = j+1$ does not hold. The assumption immediately implies $j \leq \frac{k-1}{2}$, because $k \equiv 1 \pmod{j}$. The recurrence relation for Laguerre polynomials, given in formula 18.9.1 of [48] and which states

$$L_{n+1}^{(\alpha)}(Z) = \left(-\frac{Z}{n+1} + \frac{2n+\alpha+1}{n+1} \right) L_n^{(\alpha)}(Z) - \frac{n+\alpha}{n+1} L_{n-1}^{(\alpha)}(Z), \tag{A.5}$$

implies

$$(j+1) L_{j+1}^{(2k-2j-1)}(1) = (2k-1) L_j^{(2k-2j-1)}(1) - (2k-j-1) L_{j-1}^{(2k-2j-1)}(1).$$

Combining this relation with the equality in equation (A.2) that we want to disprove, we deduce

$$(k-1) L_j^{(2k-2j-1)}(1) = (2k-j-1) L_{j-1}^{(2k-2j-1)}(1).$$

Further multiplying the last relation with the equality in equation (A.2), we obtain

$$k(k-1) \left(L_j^{(2k-2j-1)}(1) \right)^2 = (2k-j-1)(j+1)\, L_{j+1}^{(2k-2j-1)}(1)\, L_{j-1}^{(2k-2j-1)}(1). \tag{A.6}$$

The left hand side of the last expression occurs in the inequality 18.14.12 of [48], namely

$$\left(L_n^{(\alpha)}(u) \right)^2 \geq L_{n-1}^{(\alpha)}(u) L_{n+1}^{(\alpha)}(u) \qquad (u, \alpha \in \mathbb{R}_{\geq 0}).$$

Applying it we find

$$k(k-1) \left(L_j^{(2k-2j-1)}(1) \right)^2 \geq k(k-1) L_{j+1}^{(2k-2j-1)}(1) L_{j-1}^{(2k-2j-1)}(1). \tag{A.7}$$

The equality in equation (A.2), which is assumed by contradiction, and the mentioned recurrence relation (A.5) give

$$L_{j+1}^{(2k-2j-1)}(1) = \frac{k(2k-j-1)}{(k-1)(j+1)} L_{j-1}^{(2k-2j-1)}(1).$$

Therefore, the two Laguerre polynomials $L_{j+1}^{(2k-2j-1)}(1)$ and $L_{j-1}^{(2k-2j-1)}(1)$ have the same sign.[1] Thus, combining equations (A.6) and (A.7) we deduce the inequality

$$(2k-j-1)(j+1) \geq k(k-1).$$

[1] Using an argument involving a bound on the location of their zeroes, formula 18.16.12 of [48], one can show that they really have the same sign, and that they are both positive. This is not needed for our application.

Expanding the products the inequality becomes

$$j^2 - j(2k-2) + k^2 - 3k + 1 \leq 0,$$

which admits solutions for

$$k + \sqrt{k} - 1 \geq j \geq k - \sqrt{k} - 1.$$

For $k \geq 6$ the right hand side is larger than $\frac{k-1}{2}$, this is the desired contradiction to $k \neq j+1$. Summing up, we proved that if $k \geq 6$ the equality in equation (A.2) implies $k = j+1$. Consequently, by formula 18.9.13 of [48], namely

$$L_n^{(\alpha)}(Z) = L_n^{(\alpha+1)}(Z) - L_{n-1}^{(\alpha+1)}(Z),$$

we find

$$L_{j+1}^{(0)}(1) = L_j^{(1)}(1) - L_{j+1}^{(1)}(1) = 0. \tag{A.8}$$

A last application of formula (A.4) with $n = a = j+1$ and $m = 0$ shows that equation (A.8) is impossible for $j \geq 1$. While if $j = 0$ the assumption $k \geq 2$ implies $k \neq j+1$. This is the desired contradiction to the equality in equation (A.2). In the analysis above we excluded the finite number of cases given by: $k \leq 5$, $j \geq 1$, $k \equiv 1 \pmod{j}$ and $k \neq j+1$. They are the cases $(k,j) \in \{(3,1), (4,1), (5,1), (5,2)\}$, and for them the lemma is verified by direct computation of the special values occurring in formula (A.3). Specifically, for $k = 3$ and $j = 1$

$$L_1^{(3)}(1) = 3, \qquad L_2^{(3)}(1) = \frac{11}{2},$$

for $k = 4$ and $j = 1$

$$L_1^{(5)}(1) = 5, \qquad L_2^{(5)}(1) = \frac{29}{2},$$

for $k = 5$ and $j = 1$

$$L_1^{(7)}(1) = 7, \qquad L_2^{(7)}(1) = \frac{55}{2},$$

and for $k = 5$ and $j = 2$

$$L_2^{(5)}(1) = \frac{29}{2}, \qquad L_3^{(5)}(1) = \frac{191}{6}.$$

This completes the proof of the lemma. □

We can now state the claimed eigenfunction expansion.

Theorem A.2. *Let $g(u) \in \mathcal{C}_0^\infty(\mathbb{R}_{\geq 0})$ and $k \in \mathbb{Z}$, then we have*

$$\begin{aligned} g(u) = {} & \sum_{j=0}^{k-1} \frac{(2k-2j-1)}{\Gamma(2k-j)\Gamma(j+1)} \frac{W_{k,k-j-\frac{1}{2}}(2u)}{\sqrt{2u}} \int_0^\infty \frac{W_{k,k-j-\frac{1}{2}}(2v)}{\sqrt{2v}} g(v) \frac{\mathrm{d}v}{v} \\ & + \frac{1}{\pi^2} \int_0^\infty r \sinh(2\pi r) \left| \Gamma\left(-k + \frac{1}{2} + ir\right) \right|^2 \frac{W_{k,ir}(2u)}{\sqrt{2u}} \int_0^\infty \frac{W_{k,ir}(2v)}{\sqrt{2v}} g(v) \frac{\mathrm{d}v}{v} \, \mathrm{d}r. \end{aligned}$$

Proof. Let $A_{k,ir}(u)$ be a solution of the Whittaker equation with variable $u \in \mathbb{R}$ and parameters $k \in \mathbb{Z}$ and $ir \in \mathbb{C}$, where we are not assuming $r \in \mathbb{R}$. By definition, it satisfies the equation

$$\frac{\mathrm{d}^2 A_{k,ir}(u)}{\mathrm{d}u^2} + \left(-\frac{1}{4} + \frac{k}{u} + \frac{\frac{1}{4} + r^2}{u^2} \right) A_{k,ir}(u) = 0.$$

The function $B_{k,ir}(u) := \frac{1}{\sqrt{u}} A_{k,ir}(u)$ is a solution of the equation

$$\frac{\mathrm{d}^2 B_{k,ir}(u)}{\mathrm{d}u^2} + \frac{1}{u}\frac{\mathrm{d}}{\mathrm{d}u} B_{k,ir}(u) + \left(-\frac{1}{4} + \frac{k}{u} + \frac{r^2}{u^2} \right) B_{k,ir}(u) = 0.$$

Multiplying by u^2 and changing variables to $v = e^u$, we deduce that the function $C_{k,ir}(v) := B_{k,ir}(e^u) = e^{-\frac{u}{2}} A_{k,ir}(e^u)$ is a solution of the equation

$$\frac{\mathrm{d}^2 C_{k,ir}(v)}{\mathrm{d}v^2} - \left(v^2 - kv + r^2 \right) C_{k,ir}(v) = 0, \tag{A.9}$$

which is a Sturm–Liouville equation with singularities at both ends of the interval of definition. We use the theory developed, after Weyl, by Titchmarsh in [63, chapter III] to study these equations. Using formula 13.14.28 of [48], namely

$$\mathcal{W}\left(M_{k,-ir}(u), W_{k,ir}(u) \right) = -\frac{\Gamma(1-2ir)}{\Gamma\left(-k + \frac{1}{2} - ir\right)},$$

and the symmetry $W_{\kappa,\mu}(Z) = W_{\kappa,-\mu}(Z)$, the Wronskian of the solutions $e^{-\frac{u}{2}} M_{k,-ir}(e^u)$ and $e^{-\frac{u}{2}} W_{k,-ir}(e^u)$ of (A.9) is

$$\mathcal{W}\left(e^{-\frac{u}{2}} M_{k,-ir}(e^u), e^{-\frac{u}{2}} W_{k,-ir}(e^u) \right) = -\frac{\Gamma(1-2ir)}{\Gamma\left(-k + \frac{1}{2} - ir\right)}. \tag{A.10}$$

We fix for the whole proof the relation $\lambda = r^2$ with $\mathrm{Im}(r) > 0$. As customary, we will use λ or r depending on which one is most convenient. The functions

$$\begin{aligned} \theta_k(u,\lambda) := -\frac{e^{-\frac{u}{2}}\,\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1-2ir)} \Big(& M_{k,-ir}(e^u) \frac{\mathrm{d}}{\mathrm{d}u} \left(e^{-\frac{u}{2}} W_{k,-ir}(e^u) \right)_{u=0} \\ & - W_{k,-ir}(e^u) \frac{\mathrm{d}}{\mathrm{d}u} \left(e^{-\frac{u}{2}} M_{k,-ir}(e^u) \right)_{u=0} \Big), \end{aligned} \tag{A.11}$$

and

$$\phi_k(u,\lambda) := -\frac{e^{-\frac{u}{2}}\,\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1-2ir)} \Big(M_{k,-ir}(e^u) W_{k,-ir}(1) - W_{k,-ir}(e^u) M_{k,-ir}(1) \Big) \tag{A.12}$$

are solutions of equation (A.9), because they are combinations of $e^{-\frac{u}{2}} M_{k,-ir}(e^u)$ and $e^{-\frac{u}{2}} W_{k,-ir}(e^u)$. Moreover, by formula (A.10), they satisfy the conditions

$$\theta_k(0,\lambda)=1, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\theta_k(u,\lambda)\right)_{u=0}=0,$$

$$\phi_k(0,\lambda)=0, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\phi_k(u,\lambda)\right)_{u=0}=-1.$$

Using the differentation formula 13.15.20 of [48], which reads

$$\left(Z\frac{\mathrm{d}}{\mathrm{d}Z}Z\right)^n\left(e^{-\frac{Z}{2}}Z^{\kappa-1}M_{\kappa,\mu}(Z)\right)=\left(\frac{1}{2}+\mu+\kappa\right)_n e^{-\frac{Z}{2}}Z^{\kappa+n-1}M_{\kappa+n,\mu}(Z),$$

with the analogous formula (4.4.3) for the Whittaker W-function, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{-\frac{u}{2}}M_{k,-ir}(e^u)\right)_{u=0}=&\Bigg(-\frac{e^{-\frac{u}{2}}}{2}M_{k,-ir}(e^u)+\frac{e^{\frac{u}{2}}}{2}M_{k,-ir}(e^u)-ke^{-\frac{u}{2}}M_{k,-ir}(e^u)\\&+\left(k+\frac{1}{2}-ir\right)e^{-\frac{u}{2}}M_{k+1,-ir}(e^u)\Bigg)_{u=0}\\=&-kM_{k,-ir}(1)+\left(k+\frac{1}{2}-ir\right)M_{k+1,-ir}(1),\end{aligned}\tag{A.13}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{-\frac{u}{2}}W_{k,-ir}(e^u)\right)_{u=0}=&\Bigg(-\frac{e^{-\frac{u}{2}}}{2}W_{k,-ir}(e^u)+\frac{e^{\frac{u}{2}}}{2}W_{k,-ir}(e^u)-ke^{-\frac{u}{2}}W_{k,-ir}(e^u)\\&-e^{-\frac{u}{2}}W_{k+1,-ir}(e^u)\Bigg)_{u=0}\\=&-kW_{k,-ir}(1)-W_{k+1,-ir}(1).\end{aligned}\tag{A.14}$$

Furthermore, we consider the auxiliary functions

$$m_{1,k}(\lambda):=-\frac{\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{-\frac{u}{2}}M_{k,-ir}(e^u)\right)_{u=0}}{M_{k,-ir}(1)}=\frac{kM_{k,-ir}(1)-\left(k+\frac{1}{2}-ir\right)M_{k+1,-ir}(1)}{M_{k,-ir}(1)},\tag{A.15}$$

and

$$m_{2,k}(\lambda):=-\frac{\frac{\mathrm{d}}{\mathrm{d}u}\left(e^{-\frac{u}{2}}W_{k,-ir}(e^u)\right)_{u=0}}{W_{k,-ir}(1)}=\frac{kW_{k,-ir}(1)+W_{k+1,-ir}(1)}{W_{k,-ir}(1)}.\tag{A.16}$$

And we use them to define

$$\begin{aligned}\psi_{1,k}(u,\lambda):=&\,\theta_k(u,\lambda)+m_{1,k}(\lambda)\phi_k(u,\lambda)\\=&-\frac{e^{-\frac{u}{2}}\Gamma\left(-k+\frac{1}{2}-ir\right)}{\Gamma(1-2ir)}\frac{M_{k,-ir}(e^u)}{M_{k,-ir}(1)}\left(-\mathcal{W}\left(e^{-\frac{u}{2}}M_{k,-ir}(e^u),e^{-\frac{u}{2}}W_{k,-ir}(e^u)\right)\right)\\=&-\frac{e^{-\frac{u}{2}}M_{k,-ir}(e^u)}{M_{k,-ir}(1)},\end{aligned}\tag{A.17}$$

and

$$\begin{aligned}
\psi_{2,k}(u,\lambda) :=& \,\theta_k(u,\lambda) + m_{2,k}(\lambda)\phi_k(u,\lambda) \\
=& -\frac{e^{-\frac{u}{2}}\Gamma\left(-k+\frac{1}{2}-ir\right)}{\Gamma(1-2ir)} \frac{W_{k,ir}(e^u)}{W_{k,ir}(1)} \left(-\mathcal{W}\left(e^{-\frac{u}{2}}M_{k,-ir}(e^u), e^{-\frac{u}{2}}W_{k,-ir}(e^u)\right)\right) \\
=& -\frac{e^{-\frac{u}{2}}W_{k,-ir}(e^u)}{W_{k,-ir}(1)}.
\end{aligned} \tag{A.18}$$

By construction, the functions $\psi_{1,k}(u,\lambda)$ and $\psi_{2,k}(u,\lambda)$ are solutions of equation (A.9). We remark that they are indeed nothing else than a renormalization of the solutions we chose at the beginning of our construction. The auxiliary functions we constructed so far, and their mutual relations, are given for later use. By formula 13.14.14 of [48], namely

$$M_{k,\mu}(Z) = Z^{\mu+\frac{1}{2}}\left(1+O(Z)\right) \qquad (-2\mu \notin \mathbb{N}_{>0};\ Z\to 0)\,,$$

and the formula (3.4.1) for the Whittaker W-function, the functions $\psi_{1,k}(u,\lambda)$ and $\psi_{2,k}(u,\lambda)$ are in $L^2(\mathbb{R}_{<0})$ and $L^2(\mathbb{R}_{>0})$, respectively. By formula (A.10), their Wronskian is

$$\mathcal{W}\left(\psi_{1,k}(u,\lambda),\psi_{2,k}(u,\lambda)\right) = -\frac{\Gamma(1-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right) M_{k,-ir}(1)\, W_{k,-ir}(1)}.$$

We set $\widetilde{g}(u) := g\left(\frac{e^u}{2}\right) \in \mathcal{C}_0^\infty(\mathbb{R})$. According to formula (2.18.5) of [63], we define the resolvent function by

$$\begin{aligned}
\Phi_{k,g}(u,\lambda) :=& \frac{\psi_{2,k}(u,\lambda)}{\mathcal{W}\left(\psi_{1,k}(u,\lambda),\psi_{2,k}(u,\lambda)\right)} \int_{-\infty}^{u} \psi_{1,k}(v,\lambda)\widetilde{g}(v)\,\mathrm{d}v \\
&+ \frac{\psi_{1,k}(u,\lambda)}{\mathcal{W}\left(\psi_{1,k}(u,\lambda),\psi_{2,k}(u,\lambda)\right)} \int_{u}^{\infty} \psi_{2,k}(v,\lambda)\widetilde{g}(v)\mathrm{d}v \\
=& -\, e^{-\frac{u}{2}}W_{k,-ir}(e^u) \int_{-\infty}^{u} \frac{\Gamma\left(-k+\frac{1}{2}-ir\right)}{\Gamma(1-2ir)} M_{k,-ir}(e^v)e^{-\frac{v}{2}}\widetilde{g}(v)\mathrm{d}v \\
&- \frac{\Gamma\left(-k+\frac{1}{2}-ir\right)}{\Gamma(1-2ir)} e^{-\frac{u}{2}}M_{k,-ir}(e^u) \int_{u}^{\infty} W_{k,-ir}(e^v)e^{-\frac{v}{2}}\widetilde{g}(v)\,\mathrm{d}v.
\end{aligned}$$

By the generalized Sturm–Liouville expansion theorem, discussed in chapter 3 of [63] with references to sections 2.12 and 2.15, we have the representation

$$\widetilde{g}(u) = \frac{1}{2\pi i}\lim_{R\to\infty} \int_{C(R)} \Phi_{k,g}(u,\lambda)\,\mathrm{d}\lambda,$$

where the contour $C(R)$ is the union of the segments $(R-i, R+i)$ and $(-R-i,-R+i)$ with the semicircles of radius R and center $\pm i$, and it is positively oriented. Since $W_{\kappa,\mu}(Z)$ is entire in the second parameter, $M_{\kappa,\mu}(Z)$ is holomorphic in the second parameter away from the values $-2\mu \in \mathbb{N}_{>0}$, and the Γ-function is holomorphic away from real values of the argument, the integrand is holomorphic for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. For real positive λ we have a contribution of the continuous spectrum precisely as in example 4.15 of [63], i.e., that is the case $k = 0$. Namely, $W_{k,-ir}(e^u)$ is an even function of r and therefore single-valued for $\lambda \in \mathbb{R}$, but the function

$$\frac{\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1 - 2ir)} M_{k,-ir}(e^u)$$

has a branch point at $\lambda = 0$. Therefore, the contribution to the eigenfunction expansion (A.19) of the continuous spectrum is the integral of the resolvent function on a loop starting at ∞, going around 0 in the positive sense and returning to ∞. Moreover, if $k \geq 1$ there is also a finite number of poles that contribute to the expansion. They are the poles of $\Gamma\left(-k + \frac{1}{2} - ir\right)$, and are located at $\lambda_{k,j} := -\left(k - j - \frac{1}{2}\right)^2$ for $j \in \{0, ..., k-1\}$. Thus, further applying the residue theorem, we have

$$\widetilde{g}(u) = \sum_{j=0}^{k-1} \mathrm{Res}_{\lambda=\lambda_{k,j}} \Phi_{k,g}(u,\lambda) + \frac{1}{2\pi i} \int_{\infty}^{(0+)} \Phi_{k,g}(u,\lambda)\, \mathrm{d}\lambda. \tag{A.19}$$

We first examine the contribution of the continuous spectrum, it is given by

$$\begin{aligned}\int_{\infty}^{(0+)} \Phi_{k,g}(u,\lambda)\, \mathrm{d}\lambda = \int_0^\infty e^{-\frac{u}{2}} W_{k,-ir}(e^u) \int_{-\infty}^{u} \Bigg(&\frac{\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1 - 2ir)} M_{k,-ir}(e^v) \\ &- \frac{\Gamma\left(-k + \frac{1}{2} + ir\right)}{\Gamma(1 + 2ir)} M_{k,ir}(e^v) \Bigg) e^{-\frac{v}{2}} \widetilde{g}(v)\, \mathrm{d}v\, \mathrm{d}\lambda \\ + \int_0^\infty e^{-\frac{u}{2}} \Bigg(&\frac{\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1 - 2ir)} M_{k,-ir}(e^u) \\ &- \frac{\Gamma\left(-k + \frac{1}{2} + ir\right)}{\Gamma(1 + 2ir)} M_{k,ir}(e^u) \Bigg) \int_u^\infty W_{k,-ir}(e^v) e^{-\frac{v}{2}} \widetilde{g}(v)\, \mathrm{d}v\, \mathrm{d}\lambda.\end{aligned}$$

Applying the connection formula 13.14.33 of [48], which states

$$W_{\kappa,\mu}(Z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \kappa\right)} M_{\kappa,\mu}(Z) + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} M_{\kappa,-\mu}(Z),$$

with formula (3.3.9) and the functional equation of the Γ-function, we find

$$\begin{aligned}&\frac{\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(1 - 2ir)} M_{k,-ir}(e^u) - \frac{\Gamma\left(-k + \frac{1}{2} + ir\right)}{\Gamma(1 + 2ir)} M_{k,ir}(e^u) \\ &\qquad = -\frac{1}{2ir} \left(\frac{\Gamma\left(-k + \frac{1}{2} - ir\right)}{\Gamma(-2ir)} M_{k,-ir}(e^u) + \frac{\Gamma\left(-k + \frac{1}{2} + ir\right)}{\Gamma(2ir)} M_{k,ir}(e^u) \right)\end{aligned}$$

$$= -\frac{\left|\Gamma\left(-k+\frac{1}{2}+ir\right)\right|^2}{2ir\,\left|\Gamma(2ir)\right|^2} W_{k,ir}(e^u) = -\frac{\sinh(2\pi r)\left|\Gamma\left(-k+\frac{1}{2}+ir\right)\right|^2}{\pi i} W_{k,ir}(e^u).$$

Therefore, using once more the symmetry $W_{\kappa,\mu}(Z) = W_{\kappa,-\mu}(Z)$, the contribution of the continuous spectrum to the integral in (A.19) is given by

$$\int_{\infty}^{(0+)} \Phi_{k,g}(u,\lambda)\,\mathrm{d}\lambda \tag{A.20}$$

$$= -\frac{e^{-\frac{u}{2}}}{\pi i}\int_0^\infty \sinh(2\pi r)\left|\Gamma\left(-k+\frac{1}{2}+ir\right)\right|^2 W_{k,ir}(e^u)\int_{-\infty}^{\infty} W_{k,ir}(e^v)e^{-\frac{v}{2}}\widetilde{g}(v)\,\mathrm{d}v\,\mathrm{d}\lambda$$

$$= -\frac{2e^{-\frac{u}{2}}}{\pi i}\int_0^\infty r\sinh(2\pi r)\left|\Gamma\left(-k+\frac{1}{2}+ir\right)\right|^2 W_{k,ir}(e^u)\int_{-\infty}^{\infty} W_{k,ir}(e^v)e^{-\frac{v}{2}}\widetilde{g}(v)\,\mathrm{d}v\,\mathrm{d}r.$$

We now compute the residues at the points $\lambda_{k,j} = -\left(k-j-\frac{1}{2}\right)^2$. Via the relation $\lambda = r^2$ with $\mathrm{Im}(r) > 0$ they correspond to the values $r_{k,j} := i\left(k-j-\frac{1}{2}\right)$. To compute the residues of $\Phi_{k,g}(u,\lambda)$ at these poles we first need to understand the behavior of the functions $m_{1,k}(\lambda)$ and $m_{2,k}(\lambda)$ at these points.
We first examine the case $k = 1$ and $j = 0$. Specializing $k = 1$ in formulae (A.15) and (A.16), and using the recurrence relation 13.15.1 of [48], namely

$$\left(\kappa-\mu-\frac{1}{2}\right)M_{\kappa-1,\mu}(Z) + (Z-2\kappa)M_{\kappa,\mu}(Z) + \left(\kappa+\mu+\frac{1}{2}\right)M_{\kappa+1,\mu}(Z) = 0,$$

and the analogous statement for the Whittaker W-function already cited as (3.3.12), we find

$$m_{1,1}(\lambda) = \frac{M_{1,-ir}(1) - \left(\frac{3}{2}-ir\right)M_{2,-ir}(1)}{M_{1,-ir}(1)} = \left(\frac{1}{2}+ir\right)\frac{M_{0,-ir}(1)}{M_{1,-ir}(1)},$$

and

$$m_{2,1}(\lambda) = \frac{W_{1,-ir}(1) + W_{2,-ir}(1)}{W_{1,-ir}(1)} = -\left(\frac{1}{2}+ir\right)\left(\frac{1}{2}-ir\right)\frac{W_{0,-ir}(1)}{W_{1,-ir}(1)}.$$

Since $\lambda_{1,0} = -\frac{1}{4}$ corresponds to $r_{1,0} = \frac{i}{2}$, we deduce that both $m_{1,1}(\lambda)$ and $m_{2,1}(\lambda)$ have a simple zero at $\lambda = \lambda_{1,0}$ with asymptotic expansions at $\lambda_{1,0}$ given by

$$m_{1,1}(\lambda) = (\lambda-\lambda_{1,0})\frac{M_{0,\frac{1}{2}}(1)}{M_{1,\frac{1}{2}}(1)} + O\left((\lambda-\lambda_{1,0})^2\right) \qquad (\lambda\to\lambda_{1,0}), \tag{A.21}$$

and

$$m_{2,1}(\lambda) = -(\lambda-\lambda_{1,0})\frac{W_{0,\frac{1}{2}}(1)}{W_{1,\frac{1}{2}}(1)} + O\left((\lambda-\lambda_{1,0})^2\right) \qquad (\lambda\to\lambda_{1,0}), \tag{A.22}$$

respectively. We can now calculate the residue of $\Phi_{1,g}(u,\lambda)$. We are in case $ii)$ of [63, section 2.18], which states that if $m_{1,1}(\lambda) \sim a_1(\lambda - \lambda_{1,0})$ and $m_{2,1}(\lambda) \sim a_2(\lambda - \lambda_{1,0})$ for $\lambda \sim \lambda_{1,0}$ and with $a_1 \neq a_2$, then the residue of $\Phi_{1,g}(u,\lambda)$ at $\lambda_{1,0}$ is given by

$$\mathrm{Res}_{\lambda=\lambda_{1,0}}\Phi_{1,g}(u,\lambda) = \frac{\theta_1(u,\lambda_{1,0})}{a_1 - a_2} \int_{-\infty}^{\infty} \theta_1(v,\lambda_{1,0})\, \widetilde{g}(v)\mathrm{d}v.$$

By equations (A.21) and (A.22) we obtain

$$\mathrm{Res}_{\lambda=\lambda_{1,0}}\Phi_{1,g}(u,\lambda) = \frac{\theta_1(u,\lambda_{1,0})}{\frac{M_{0,\frac{1}{2}}(1)}{M_{1,\frac{1}{2}}(1)} + \frac{W_{0,\frac{1}{2}}(1)}{W_{1,\frac{1}{2}}(1)}} \int_{-\infty}^{\infty} \theta_1(v,\lambda_{1,0})\, \widetilde{g}(v)\mathrm{d}v.$$

We quote the special cases of Whittaker functions given in formulae 13.18.1 and 13.18.2 of [48], they are

$$M_{0,\frac{1}{2}}(v) = 2\sinh\left(\frac{v}{2}\right), \qquad W_{0,\frac{1}{2}}(v) = e^{-\frac{v}{2}}, \qquad M_{1,\frac{1}{2}}(v) = W_{1,\frac{1}{2}}(v) = v\, e^{-\frac{v}{2}}.$$

Using them, we compute

$$\frac{M_{0,\frac{1}{2}}(1)}{M_{1,\frac{1}{2}}(1)} + \frac{W_{0,\frac{1}{2}}(1)}{W_{1,\frac{1}{2}}(1)} = \sqrt{e}\left(\sqrt{e} - \frac{1}{\sqrt{e}}\right) + \frac{\sqrt{e}}{\sqrt{e}} = e.$$

Since $m_{2,1}(\lambda_{1,0}) = 0$, formula (A.18) implies

$$\theta_1(u,\lambda_{1,0}) = \psi_{2,1}(\lambda_{1,0}) = -\frac{e^{-\frac{u}{2}} W_{1,\frac{1}{2}}(e^u)}{W_{1,\frac{1}{2}}(1)} = -e^{-\frac{u}{2}+\frac{1}{2}} W_{1,\frac{1}{2}}(e^u),$$

where in the last equality we used $W_{1,\frac{1}{2}}(1) = e^{-\frac{1}{2}}$. Therefore, the expression of the residue simplifies to

$$\mathrm{Res}_{\lambda=\lambda_{1,0}}\Phi_{1,g}(u,\lambda) = e^{-\frac{u}{2}} W_{1,\frac{1}{2}}(e^u) \int_{-\infty}^{\infty} e^{-\frac{v}{2}}\, W_{1,\frac{1}{2}}(e^v)\, \widetilde{g}(v)\, \mathrm{d}v. \tag{A.23}$$

We now compute the residues in the case $k \geq 2$ and $j \in \{0, ..., k-1\}$. We are now going to show that both $m_{1,k}(\lambda)$ and $m_{2,k}(\lambda)$ are non-zero and holomorphic at each $\lambda_{k,j}$. We first prove the claim for $m_{2,k}(\lambda)$, the statement for $m_{1,k}(\lambda)$ will follow from this and general considerations. Equation (A.16) states

$$m_{2,k}(\lambda) = \frac{kW_{k,-ir}(1) + W_{k+1,-ir}(1)}{W_{k,-ir}(1)}.$$

Since $W_{k,\mu}(z)$ is entire in μ, it is enough to prove that nor the numerator nor the denominator of the last expression are zero at the points $\lambda_{k,j} = -\left(k - j - \frac{1}{2}\right)^2$ corresponding to

$r_{k,j} = i\left(k-j-\frac{1}{2}\right)$. We express them in terms of generalized Laguerre polynomials via formula (3.2.5), to obtain

$$W_{k,k-j-\frac{1}{2}}(1) = \frac{(-1)^j\, j!}{\sqrt{e}} L_j^{(2k-2j-1)}(1),$$
$$W_{k+1,k-j-\frac{1}{2}}(1) = \frac{(-1)^{j+1}\,(j+1)!}{\sqrt{e}} L_{j+1}^{(2k-2j-1)}(1).$$

Therefore, we find

$$m_{2,k}(\lambda_{k,j}) = \frac{kL_j^{(2k-2j-1)}(1) - (j+1)L_{j+1}^{(2k-2j-1)}(1)}{L_j^{(2k-2j-1)}(1)}.$$

Applying lemma A.1, we deduce that $m_{2,k}(\lambda)$ is finite and non-zero at each $\lambda_{k,j}$, as claimed. Regarding the function $m_{1,k}(\lambda)$, by construction we have

$$m_{1,k}(\lambda) - m_{2,k}(\lambda) = \mathcal{W}\left(\psi_{1,k}(u,\lambda), \psi_{2,k}(u,\lambda)\right) = -\frac{\Gamma(1-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right) M_{k,-ir}(1)\, W_{k,-ir}(1)}.$$

Using formula 13.18.17 of [48], namely

$$M_{\frac{\alpha+1}{2}+n,\frac{\alpha}{2}}(Z) = \frac{e^{-\frac{Z}{2}} Z^{\frac{\alpha+1}{2}} \Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)} L_n^{(\alpha)}(Z) \qquad (n \in \mathbb{N}),$$

and the analogous statement (3.2.5) for the Whittaker W-function, we obtain

$$\begin{aligned} m_{1,k}(\lambda_{k,j}) - m_{2,k}(\lambda_{k,j}) =& \frac{\Gamma(1-2ir_{k,j})}{\Gamma\left(-k+\frac{1}{2}-ir_{k,j}\right) M_{k,-ir_{k,j}}(1)\, W_{k,-ir_{k,j}}(1)} \\ =& \frac{\Gamma(2k-2j)}{\Gamma(-j)\, M_{k,k-j-\frac{1}{2}}(1)\, W_{k,k-j-\frac{1}{2}}(1)} \\ =& \frac{(-1)^j}{e} \frac{\Gamma(2k-j-1)}{\Gamma(-j)\,\Gamma(j+1)^2 \left(L_j^{(2k-2j-1)}(1)\right)^2}. \end{aligned}$$

Further using equation (A.1) and the assumption $j \in \{0, ..., k-1\}$ we see that the function $m_{1,k}(\lambda) - m_{2,k}(\lambda)$ has a simple zero at every $\lambda_{k,j}$. Thus, it must hold $m_{1,k}(\lambda_{k,j}) = m_{2,k}(\lambda_{k,j})$. This completes the claim that also $m_{1,k}(\lambda)$ is non-zero at each $\lambda_{k,j}$.

We just showed that for $k \geq 2$ we are in case i) of section 2.18 of [63]. It states that if $m_{1,k}(\lambda_{k,j}) = m_{2,k}(\lambda_{k,j}) = a \neq 0$, and $m_{1,k}(\lambda) - m_{2,k}(\lambda) \sim b\,(\lambda - \lambda_{k,j})$, then the residue of $\Phi_{k,g}(u,\lambda)$ at $\lambda = \lambda_{k,j}$ is given by the formula

$$\mathrm{Res}_{\lambda=\lambda_{k,j}} \Phi_{k,g}(u,\lambda) = \frac{\theta_k(u,\lambda_{k,j}) + a\phi_k(u,\lambda_{k,j})}{b} \int_{-\infty}^{\infty} \theta_k(v,\lambda_{k,j}) + a\phi_k(v,\lambda_{k,j})\, \widetilde{g}(v)\, \mathrm{d}v.$$

Since $\lambda - \lambda_{k,j} = (ir - ir_{k,j})(ir + ir_{k,j})$, we compute

$$\begin{aligned}
\mathrm{Res}_{\lambda=\lambda_{k,j}}\Gamma\left(-k+\frac{1}{2}-ir\right) &= \lim_{\lambda\to\lambda_{k,j}}(\lambda-\lambda_{k,j})\Gamma\left(-k+\frac{1}{2}-ir\right)\\
&= \lim_{ir\to ir_{k,j}}(ir-ir_{k,j})(ir+ir_{k,j})\Gamma\left(-k+\frac{1}{2}-ir\right)\\
&= \mathrm{Res}_{ir=ir_{k,j}}\Gamma\left(-k+\frac{1}{2}-ir\right)\cdot 2ir_{k,j} = -\frac{(-1)^j(2k-2j-1)}{\Gamma(j+1)},
\end{aligned}$$

which implies

$$\begin{aligned}
m_{1,k}(\lambda)-m_{2,k}(\lambda) &= -\frac{\Gamma(1-2ir)}{\Gamma\left(-k+\frac{1}{2}-ir\right)M_{k,-ir}(1)\,W_{k,ir}(1)}\\
&= (\lambda-\lambda_{k,j})\,\frac{(-1)^j\,\Gamma(j+1)\,\Gamma(2k-2j)}{(2k-2j-1)\,M_{k,k-j-\frac{1}{2}}(1)\,W_{k,k-j-\frac{1}{2}}(1)}\\
&\quad + O\left((\lambda-\lambda_{k,j})^2\right) \qquad (\lambda\to\lambda_{k,j})\,.
\end{aligned}$$

From formula 13.18.17 of [48], namely

$$W_{\frac{\alpha+1}{2}+n,\frac{\alpha}{2}}(Z) = (-1)^n\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}M_{\frac{\alpha+1}{2}+n,\frac{\alpha}{2}}(Z) \qquad (n\in\mathbb{N})\,,$$

we obtain the relation

$$M_{k,k-j-\frac{1}{2}}(1) = (-1)^j\frac{\Gamma(2k-2j)}{\Gamma(2k-j)}W_{k,k-j-\frac{1}{2}}(1).$$

The equality $m_{1,k}(\lambda_{k,j}) = m_{2,k}(\lambda_{k,j})$ applied to formulae (A.17) and (A.18) implies

$$\begin{aligned}
\theta_k(u,\lambda_{k,j}) + m_{1,k}(\lambda_{k,j})\phi_k(u,\lambda_{k,j}) &= \theta_k(u,\lambda_{k,j}) + m_{2,k}(\lambda_{k,j})\phi_k(u,\lambda_{k,j})\\
&= \psi_{1,k}(u,\lambda_{k,j}) = \psi_{2,k}(u,\lambda_{k,j}).
\end{aligned}$$

Using these relations, the residue of $\Phi_{k,g}(u,\lambda)$ at $\lambda_{k,j}$ is given by the expression

$$\mathrm{Res}_{\lambda=\lambda_{k,j}}\Phi_{k,g}(u,\lambda) = \frac{(2k-2j-1)}{\Gamma(2k-j)\Gamma(j+1)}e^{-\frac{u}{2}}\,W_{k,k-j-\frac{1}{2}}(e^u)\int_{-\infty}^{\infty} W_{k,k-j-\frac{1}{2}}(e^v)\,e^{-\frac{v}{2}}\,\widetilde{g}(v)\,\mathrm{d}v. \tag{A.24}$$

We observe that, even though the proof is different, the statement of equation (A.23) is formally obtained by equation (A.24) setting $k=1$ and $j=0$. Combining equations (A.19), (A.20), (A.23) and (A.24), we find

$$\begin{aligned}
\widetilde{g}(u) &= \sum_{j=0}^{k-1}\frac{(2k-2j-1)}{\Gamma(2k-j)\Gamma(j+1)}\,e^{-\frac{u}{2}}\,W_{k,k-j-\frac{1}{2}}(e^u)\int_{-\infty}^{\infty} W_{k,k-j-\frac{1}{2}}(e^v)\,e^{-\frac{v}{2}}\,\widetilde{g}(v)\,\mathrm{d}v\\
&\quad+\frac{1}{\pi^2}\int_0^{\infty} r\sinh(2\pi r)\left|\Gamma\left(-k+\frac{1}{2}+ir\right)\right|^2 e^{-\frac{u}{2}}W_{k,ir}(e^u)\int_{-\infty}^{\infty}W_{k,ir}(e^v)e^{-\frac{v}{2}}\,\widetilde{g}(v)\,\mathrm{d}v\,\mathrm{d}r.
\end{aligned}$$

Recalling the assignment $\widetilde{g}(u) = g\left(\frac{e^u}{2}\right)$ and changing variables to $a = \frac{e^u}{2}$ and $b = \frac{e^v}{2}$ proves

$$
\begin{aligned}
g(a) = &\sum_{j=0}^{k-1} \frac{(2k-2j-1)}{\Gamma(2k-j)\Gamma(j+1)} \frac{W_{k,k-j-\frac{1}{2}}(2a)}{\sqrt{2a}} \int_0^\infty \frac{W_{k,k-j-\frac{1}{2}}(2b)}{\sqrt{2b}} \, g(b) \, \frac{\mathrm{d}b}{b} \\
&+ \frac{1}{\pi^2} \int_0^\infty r \sinh(2\pi r) \left| \Gamma\left(-k+\frac{1}{2}+ir\right) \right|^2 e^{-\frac{u}{2}} \frac{W_{k,ir}(2a)}{\sqrt{2a}} \int_0^\infty \frac{W_{k,ir}(2b)}{\sqrt{2b}} g(b) \, \frac{\mathrm{d}b}{b} \, \mathrm{d}r.
\end{aligned}
$$

The proof of the theorem is complete. $\square$

182

Appendix B

Mellin Transform and Special Functions

In this appendix we collect definition and relevant properties of the Mellin transform and of the special functions that occur in the thesis, and we give references for further properties of the objects under consideration. We would like to explicitly mention the excellent source [48], which is the printed counterpart of the Digital Library of Mathematical Functions [47].

The Mellin transform is an integral transform, discussed in details in [20, chapter 3] and [62, chapter 9], related to the Fourier and Laplace transforms.

Definition B.1. Let $f : \mathbb{R}_{>0} \to \mathbb{C}$ be a locally Lebesgue integrable function with asymptotic expansions

$$f(t) = \begin{cases} O(t^\alpha), & t \to 0, \\ O(t^\beta), & t \to \infty, \end{cases}$$

with $\alpha > \beta$. Then its *Mellin transform* is the function defined by

$$\mathcal{M}\left(f(t), s\right) = \int_0^\infty f(t) t^{s-1} \mathrm{d}t \qquad \left(-\alpha < \mathrm{Re}(s) < -\beta\right),$$

and by analytic continuation.

Observe that $\mathcal{M}\left(f(t), s\right)$ is holomorphic in the strip of definition of its integral representation. An important feature of the Mellin transform is the relation between the asymptotic expansions of $f(t)$ for t small and large and the poles of $\mathcal{M}\left(f(t), s\right)$. One direction of this relation is given by the next theorem.

Theorem B.2. *Let $f(t)$ be locally Lebesgue integrable with the asymptotic expansions*

$$f(t) = \sum_{j=1}^{n} c_j t^{\alpha_j} + O\big(t^A\big) \qquad \left(t \to 0,\, c_j \in \mathbb{C},\, -A < -\alpha_j \leq -\alpha\right),$$

$$f(t) = \sum_{l=1}^{m} d_l t^{\beta_l} + O\big(t^B\big) \qquad \left(t \to \infty,\, d_l \in \mathbb{C},\, -\beta \leq -\beta_l \leq -B\right).$$

Then $\mathcal{M}\left(f(t), s\right)$, which is holomorphic in the strip

$$\left\{ s \in \mathbb{C} \,|\, \mathrm{Re}(s) \in (-\alpha, -\beta) \right\},$$

has a meromorphic continuation to the strip

$$\{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \in (-A, -B)\} \,.$$

Furthermore, in a neighborhood of $-\alpha_j$, it has the expansion

$$\mathcal{M}\left(f(t), s\right) = \frac{c_j}{s+\alpha_j} + O(1).$$

Proof. We restrict the proof to the extension of the strip of definition to the left, corresponding to the asymptotic expansion for t small. Consider the auxiliary function

$$g(t) := f(t) - \sum_{j=1}^{n} c_j t^{\alpha_j}.$$

Then, we have

$$\mathcal{M}\left(f(t), s\right) = \int_0^1 g(t) t^{s-1} \mathrm{d}t + \int_0^1 \sum_{j=1}^{n} c_j t^{\alpha_j} t^{s-1} \mathrm{d}t + \int_1^\infty f(t) t^{s-1} \mathrm{d}t.$$

Since $g(t)$ is $O\big(t^A\big)$ for $t \to 0$, the first integral is defined for $\mathrm{Re}(s) > -A$. Moreover, the third integral is well-defined for $\mathrm{Re}(s) < -\beta$. Specifically, in a neighborhood of $s = -\alpha_j$ they can be regarded as constant functions. Explicitly integrating we find the relation

$$\int_0^1 t^{\alpha_j} t^{s-1} \mathrm{d}t = \frac{1}{s+\alpha_j} \qquad (\mathrm{Re}(s) > -\alpha_j)\,,$$

whose right hand side gives the analytic continuation of the second integral to the whole complex plane. Summing the contributions the result is proven. □

The references cited in the opening the section also state an inverse mapping theorem, which establishes the asymptotic expansions of $f(t)$ from the poles of $\mathcal{M}\left(f(t), s\right)$. Under suitable conditions on $f(t)$, the Mellin transform can be inverted via the formula

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{M}\left(f(t), s\right) t^{-s} \mathrm{d}s,$$

where c lies in the strip of holomorphicity of $\mathcal{M}\left(f(t), s\right)$. The behavior of the Mellin transform under elementary manipulations is given by the following relations, which are valid for s ensuring the well-definition of $\mathcal{M}\left(f(t), s\right)$:

(1) $\mathcal{M}\left(t^h f(t), s\right) = \mathcal{M}\left(f(t), s+h\right) \qquad (h \in \mathbb{C})\,.$

(2) $\mathcal{M}\left(\sum_{j\in J} \mu_j f(\lambda_j t), s\right) = \mathcal{M}\left(f(t), s\right) \cdot \left(\sum_{j\in J} \mu_j \lambda_j^{-s}\right) \qquad (\mu_j, \lambda_j \in \mathbb{C},\, |J| < \infty)\,.$

(3) $\frac{\mathrm{d}}{\mathrm{d}s} \mathcal{M}\left(f(t), s\right) = \mathcal{M}\left(f(t)\log(t), s\right).$

(4) $\mathcal{M}\left(\frac{\mathrm{d}}{\mathrm{d}t}f(t), s\right) = -(s-1)\mathcal{M}\left(f(t), s-1\right).$

(5) $\mathcal{M}\left(\int_0^t f(u)\mathrm{d}u, s\right) = -\frac{1}{s}\mathcal{M}\left(f(t), s+1\right).$

Moreover, we can explicit the relation with other important integral transforms: Let $\mathcal{F}$ be the Fourier transform and $\mathcal{L}$ be the Laplace transform, then it holds

$$\mathcal{M}\left(f(t), s\right) = \mathcal{F}\left(f\left(e^t\right), is\right) = \mathcal{L}\left(f\left(e^t\right), -s\right) + \mathcal{L}\left(f\left(e^t\right), s\right).$$

Following a note of Zagier [69], we observe that, adapting the argument that proves the direct mapping theorem B.2, it is possible to define the Mellin transform of a function $f(t)$ with asymptotic expansions of the form

$$f(t) = \begin{cases} \sum_{j=1}^{n} c_j t^{\alpha_j} + O\left(t^A\right), & t \to 0, \\ \sum_{l=1}^{m} d_j t^{\beta_l} + O\left(t^B\right), & t \to \infty, \end{cases}$$

with $\alpha_1 < ... < \alpha_n < A$, $\beta_1 > ... > \beta_m > B$ and $A > B$. If we further assume that $\alpha_1 > \beta_1$ we are in the hypothesis of definition B.1. If this is not the case, the integral

$$\int_0^\infty f(t) t^{s-1} \mathrm{d}t$$

is not convergent for any $s \in \mathbb{C}$. In this case the Mellin transform is defined using the linearity of the integral: Let

$$\mathcal{M}\left(f(t), s\right) := \int_0^T f(t) t^{s-1} \mathrm{d}t + \int_T^\infty f(t) t^{s-1} \mathrm{d}t \qquad (T \in \mathbb{R}_{>0}).$$

The first term is holomorphic for $\mathrm{Re}(s) > -\alpha_1$ and, via the argument of the direct mapping theorem, it can be analytically continued to $\mathrm{Re}(s) > -A$. Similarly the second term defines a meromorphic function for $\mathrm{Re}(s) < -B$. Their sum, clearly independent of the choice of T, is a meromorphic function in the strip

$$\{s \in \mathbb{C} \,|\, \mathrm{Re}(s) \in (-A, -B)\}.$$

This shows the well-definition of this generalized Mellin transform, to which we still refer as the Mellin transform.

Example B.3. Let $f(t) = t^u$ for $u \in \mathbb{R}$, its Mellin transform is

$$\mathcal{M}\left(t^u, s\right) = \int_0^1 t^{u+s-1} \mathrm{d}t + \int_1^\infty t^{u+s-1} \mathrm{d}t.$$

The first integral is analytically continued by the relation

$$\int_0^1 t^{u+s-1} \mathrm{d}t = \frac{1}{u+s} \qquad (\mathrm{Re}(s) > -u),$$

and the second one by the relation

$$\int_1^{\infty} t^{u+s-1} \mathrm{d}t = -\frac{1}{u+s} \qquad (\mathrm{Re}(s) < -u)\,.$$

Therefore the Mellin transform of t^u is identically zero on the whole complex plane.

Remark B.4. By the last example, we have the following equality of meromorphic functions

$$\mathcal{M}\left(f(t), s\right) = \mathcal{M}\left(f(t) + \sum_{j=0}^{n} a_j t^{u_j}, s\right) \qquad (n \in \mathbb{N};\ a_j \in \mathbb{C};\ u_j \in \mathbb{R})\,,$$

which is valid on the domain of definition of any of the two Mellin transforms.

We now proceed to define and describe the elementary properties of the special functions occurring in this work. The Γ-*function*, discussed in chapter 5 of [48] together with its variations, is defined to be the Mellin transform of the exponential e^{-t}

$$\Gamma(Z) := \mathcal{M}\left(e^{-t}, Z\right).$$

The Γ-function is meromorphic on the whole complex plane with simple poles at $-n$ for $n \in \mathbb{N}$, and corresponding residues

$$\mathrm{Res}_{Z=-n}\left(\Gamma(Z)\right) = \frac{(-1)^n}{n!} \qquad (n \in \mathbb{N})\,.$$

Moreover, it satisfies the functional equation

$$\Gamma(Z+1) = Z\,\Gamma(Z).$$

The *Pochhammer symbols* are a special notation for notable quotients of Γ-functions, namely

$$(Z)_n := \frac{\Gamma(Z+n)}{\Gamma(Z)} \qquad (n \in \mathbb{N})\,.$$

The *Digamma function* is the logarithmic derivative of the Γ-function

$$\psi_0(Z) := \frac{\frac{\mathrm{d}}{\mathrm{d}Z}\Gamma(Z)}{\Gamma(Z)} \qquad (-Z \notin \mathbb{N})\,.$$

Also related to the Γ-function is the *Barnes G-function*, whose most complete reference is probably still the original work of Barnes in [5], and which is defined by the product form

$$G(Z+1) = (2\pi)^{\frac{Z}{2}} e^{-\frac{Z+Z^2(1+\gamma)}{2}} \prod_{k=1}^{\infty} \left(\left(1+\frac{Z}{k}\right)^k e^{\frac{Z^2}{2k}-Z}\right),$$

where $\gamma := -\psi_0(1)$ is the *Euler–Mascheroni constant*. The Barnes G-function satisfies the conditions

$$G(1) = 1, \qquad G(Z+1) = \Gamma(Z)G(Z),$$

and has therefore the special values

$$G(n) = \prod_{j=0}^{n-2} j! \qquad (n \in \mathbb{N}_{\geq 2})\,.$$

The *Riemann ζ-function*, to which is dedicated chapter 25 of [48] and a vast literature, is defined by the formula

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (\mathrm{Re}(s) > 1)\,,$$

and it is extended by analytic continuation to a meromorphic function on the whole complex plane whose only singularity is a simple pole at $s = 1$. In this work appear the special values

$$\begin{aligned}\zeta(-1) &= -\frac{1}{12}\\ \zeta'(-1) &= \frac{1}{12} - \log\left(\lim_{n\to\infty} \frac{\prod_{j=1}^{n} j^j}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}}\right).\end{aligned}$$

A direct generalization iof the Riemann ζ-function is the *Hurwitz ζ-function*, defined by the formula

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\mathrm{Re}(s) > 1,\, -a \notin \mathbb{N})\,,$$

and extended by analytic continuation to a meromorphic function on the whole complex plane. It is related to the Digamma function by the relation

$$\frac{\mathrm{d}}{\mathrm{d}Z}\psi_0(Z) = \zeta(2, Z) \qquad (-z \notin \mathbb{N})\,.$$

In the context of ζ-functions we also discuss the Selberg $\mathcal{Z}$-function, which is discussed in great detail in chapter 2 of [33] and more informally in paragraph 7 of [35]. Let Γ be a cofinite and discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. An element $\gamma \in \Gamma$ is hyperbolic if $|\mathrm{tr}(\gamma)| > 2$, and it is primitive if it generates its own centralizer in Γ. Proceeding as in the work, let $H(\Gamma)$ be a maximal set of inconjugate primitive hyperbolic elements, fixed in such a way that if γ is a chosen element also γ^{-1} is. Moreover, define the norm $N(\gamma)$ of a hyperbolic element by the relation

$$N(\gamma)^{\frac{1}{2}} + N(\gamma)^{-\frac{1}{2}} = |\mathrm{tr}(\gamma)|\,,$$

which is related to the length

$$\ell(\gamma) = \inf_{z\in\mathbb{H}} d_{\mathrm{hyp}}(\gamma(z), z)$$

of theassociated closed geodesic on $\Gamma\backslash\mathbb{H}$ by the relation

$$N(\gamma) = e^{\ell(\gamma)}.$$

The Selberg $\mathcal{Z}$-function $\mathcal{Z}_\Gamma(s)$ associated to Γ is defined by the formula

$$\mathcal{Z}_\Gamma(s) := \prod_{\gamma\in H(\Gamma)} \prod_{n=0}^{\infty} \left(1 - N(\gamma)^{-s-n}\right) = \prod_{\gamma\in H(\Gamma)} \prod_{n=0}^{\infty} \left(1 - e^{-\ell(\gamma)(s+n)}\right) \qquad (\mathrm{Re}(s) > 1)\,,$$

and it is extended to a meromorphic function on the whole complex plane by analytic continuation. The Selberg $\mathcal{Z}$-function is an entire function with a simple zero at $s = 1$ and no other zeroes at the positive integers.

In the work we use two families of orthogonal polynomials, described in chapter 18 of [48]. The *Chebyshev polynomials of the first kind* $\{T_n(x)\}_{n\in\mathbb{Z}}$ are polynomials with integer coefficients of degree $|n|$ which are defined by the orthogonality relation

$$\int_{-1}^{1} T_n(u)T_m(u)\frac{\mathrm{d}u}{\sqrt{1-u^2}} = 0 \qquad (n, m \in \mathbb{N},\, n \neq m)\,,$$

the normalization

$$\int_{-1}^{1} (T_n(u))^2 \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \begin{cases} \frac{\pi}{2}, & n > 0, \\ \pi, & n = 0, \end{cases}$$

and the symmetry relation

$$T_n(u) = T_{-n}(u).$$

We also use the explicit formula

$$T_n(u) = \cosh(n \operatorname{arccosh}(u)) \qquad (u \geq 1)\,.$$

For $\alpha \in \mathbb{C}$, the *generalized Laguerre polynomials* $\left\{L_n^{(\alpha)}(x)\right\}_{n\in\mathbb{N}}$ are polynomials with rational coefficients of degree n defined by the orthogonality relation

$$\int_0^{\infty} L_n^{(\alpha)}(u)L_m^{(\alpha)}(u)e^{-u}u^{\alpha}\mathrm{d}u = 0 \qquad (n \neq m),$$

and the normalization

$$\int_0^{\infty} \left(L_n^{(\alpha)}(u)\right)^2 e^{-u}u^{\alpha}\mathrm{d}u = (\alpha)_{n+1}.$$

They are given by the explicit formula

$$L_n^{(\alpha)}(Z) = \sum_{l=0}^{n} \frac{(\alpha+l+1)_{n-l}}{\Gamma(n-l+1)\Gamma(l+1)}(-Z)^l.$$

The modified Bessel equation, for which we refer to chapter 10 of [48],

$$Z^2\frac{\mathrm{d}^2w}{\mathrm{d}Z^2} + Z\frac{\mathrm{d}w}{\mathrm{d}Z} - (Z^2+\nu^2)w = 0 \qquad (\nu \in \mathbb{C})$$

has, for $|\arg(Z)| \leq \frac{\pi}{2}$ and $\mathrm{Re}(\nu) > 0$, the pair of standard solutions given by the *modified Bessel function of the first kind*

$$I_\nu(Z) := \left(\frac{Z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(\frac{Z}{2}\right)^{2n}}{\Gamma(\nu+n+1)\Gamma(n+1)}$$

and the *modified Bessel function of the second kind* $K_\nu(Z)$, whose defining property is the asymptotic expansion

$$K_\nu(Z) \sim \sqrt{\frac{\pi}{2Z}}e^{-Z} \qquad \left(|\arg(Z)| < \frac{3\pi}{2},\ |Z| \to \infty\right).$$

Both $I_\nu(Z)$ and $K_\nu(Z)$ have a branch point at $Z = 0$ for every $\nu \in \mathbb{C}$, and, for fixed $Z \neq 0$, each of their branches is entire for $\nu \in \mathbb{C}$. Moreover, $I_\nu(Z)$, $K_\nu(Z)$ and $K_{i\nu}(Z)$ are real valued for $\nu, Z \in \mathbb{R}$.
Similarly, the Whittaker equation, for which we refere to chapter 10 of [48] or to [10],

$$\frac{\mathrm{d}^2w}{\mathrm{d}Z^2} + \left(-\frac{1}{4} + \frac{\kappa}{Z} + \frac{\frac{1}{4}-\mu^2}{Z^2}\right)w = 0 \qquad (\kappa, \mu \in \mathbb{C})$$

has as standard solutions the *Whittaker M-function*

$$M_{\kappa,\mu}(Z) := e^{-\frac{Z}{2}}Z^{\frac{1}{2}+\mu}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+\mu-\kappa\right)_n}{(1+2\mu)_n}\frac{Z^n}{\Gamma(n+1)} \qquad (2\mu \notin \mathbb{Z}_{<0}),$$

and the *Whittaker W-function* $W_{\kappa,\mu}(z)$, whose defining property is the expansion

$$W_{\kappa,\mu}(Z) \sim e^{-\frac{Z}{2}}Z^\kappa \qquad \left(|\arg(Z)| < \frac{3\pi}{2},\ |Z| \to \infty\right).$$

Even though $M_{\kappa,\mu}(z)$ does not exist for $2\mu \in \mathbb{Z}_{<0}$, the function $\frac{M_{\kappa,\mu}(Z)}{\Gamma(1+2\mu)}$ can be extended to these values by continuity. Both $M_{\kappa,\mu}(Z)$ and $W_{\kappa,\mu}(Z)$ have branch points at $Z = 0, \infty$ for every $\nu \in \mathbb{C}$, and, for fixed $Z \neq 0$, each branch of $\frac{M_{\kappa,\mu}(Z)}{\Gamma(1+2\mu)}$ and $W_{\kappa,\mu}(Z)$ is entire in κ and μ. Moreover, both Whittaker functions are real valued for $\kappa, Z \in \mathbb{R}$. Modified Bessel functions and generalized Laguerre polynomials are special cases of Whittaker functions, according to the formulae

$$I_\nu(Z) = \frac{2^{-2\nu-\frac{1}{2}}}{\sqrt{Z}\,\Gamma(1+\nu)}M_{0,\nu}(2Z),$$

$$K_\nu(Z) = \sqrt{\frac{\pi}{2Z}} W_{0,\nu}(2Z),$$
$$L_n^{(\alpha)}(Z) = \frac{(-1)^n Z^{-\frac{\alpha+1}{2}} e^{\frac{Z}{2}}}{\Gamma(n+1)} W_{\frac{\alpha+1}{2}+n,\frac{\alpha}{2}}(Z).$$

The *generalized hypergeometric functions* are a fundamental class of special functions, described in chapters 15 and 16 of [48]. Let $p, q \in \mathbb{N}$ and $a_1, ..., a_p, b_1, ..., b_q \in \mathbb{C}$, then the generalized hypergeometric function ${}_pF_q\left(a_1, ..., a_p; b_1, ..., b_q; z\right)$ is formally defined by the series

$${}_pF_q\left(a_1, ..., a_p; b_1, ..., b_q; Z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{Z^n}{\Gamma(n+1)}.$$

If $p \leq q$ the series on the right hand side defines an entire function on $\mathbb{C}$. If $p = q+1$ there are two possible cases: If one of the a_j is a non-positive integer, then the series terminates and defines a complex polynomial, thus an entire function. If none of the a_j is a non-positive integer then the series converges in the disk $|Z| < 1$, and outside the disk the hypergeometric function is defined by analytic continuation. We do not discuss the degenerate case $p > q+1$.
In this work we make use of two classical formulae to determine special values of the generalized hypergeometric function. The Chu–Vandermonde identity[1] states

$${}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n} \qquad (n \in \mathbb{N}).$$

And Saalschütz's theorem states

$${}_3F_2(a, b, -n; c, 1+a+b-c-n; 1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \qquad (n \in \mathbb{N}).$$

We also note that ${}_2F_1(a, b; c; Z)$ is a solution of the hypergeometric equation

$$Z(1-Z)\frac{\mathrm{d}^2 w}{\mathrm{d}Z^2} + (c - (a+b+1)Z)\frac{\mathrm{d}w}{\mathrm{d}Z} - ab\, w = 0,$$

which has the modified Bessel equation and the Whittaker equations as degenerate forms.

The *modified Legendre function*, discussed in [23], is defined in terms of the hypergeometric function

$$P_{s,k}(u) := \left(\frac{2}{1+u}\right)^s {}_2F_1\left(s-k, s+k; 1; \frac{u-1}{u+1}\right) \qquad (s \in \mathbb{C};\ k \in \mathbb{Z};\ u \in \mathbb{R}_{\geq 0}).$$

Finally, the *elliptic integrals* $F(\phi, l)$, for which we refer to chapter 19 of [48], are defined by the formula

$$F(\phi, l) := \int_0^\phi \frac{\mathrm{d}\theta}{\sqrt{1 - l^2 \sin(\theta)^2}} \qquad \left(1 - \sin(\phi)^2, 1 - l^2 \sin(\phi)^2 \notin \mathbb{R}_{\leq 0}\right).$$

[1]The Chu–Vandermonde identity is indeed a very old result. According to page 59 of [4] its first appearance in modern european mathematics has been in 1772 by Vandermonde [64], even though it is possible that Euler had it even before. But the result had been published in China almost five hundred years before by Chu [16].

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Selbständigkeitserklärung

Hiermit versichere ich, dass ich die vorgelegte Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe.

Ich erkläre, dass ich die Arbeit erstmalig und nur an der Humboldt-Universität zu Berlin eingereicht habe und mich nicht anderwärts um einen Doktorgrad beworben habe. In dem Promotionsfach besitze ich keinen Doktorgrad.

Der Inhalt der dem Verfahren zugrundeliegenden Promotionsordnung ist mir bekannt.

Berlin, den 3. Mai 2017 Giovanni De Gaetano